PHYS 813: Statistical Mechanics and Thermodynamics
Exam II

April 28, 2008, 75 minutes, closed book.

Please start each solution on a fresh sheet of paper, use only one side of the paper. Try to show how well you understand the problems. Always justify your reasoning. Assemble your solutions in the increasing numerical order. Derive all thermodynamic formulas beyond definitions of thermodynamic functions and Maxwell's relations.

1. Two-dimensional classical gas with "Coulomb" interactions consists of a mixture of N positively and N negatively charged particles of mass m in a square of side L. The Hamiltonian of the system is

$$\mathcal{H} = \sum_{i=1}^{2N} \frac{\boldsymbol{p}_i^2}{2m} - \sum_{i<j}^{2N} c_i c_j \ln |\boldsymbol{q}_i - \boldsymbol{q}_j|,$$

where $c_i = c_0$ for $i = 1, \cdots, N$ and $c_i = -c_0$ for $i = N+1, \cdots, 2N$ denote the charges of particles whereas $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$ are the coordinates and the momenta of the particles, respectively. Note that there are N^2 pairs with interactions of opposite charges and $N(N-1)$ with like charges.

   (a) Write down the canonical partition function $Q_{2N}(T, A)$ where T is the temperature and A is the area of the square. Perform the integrals over the momenta.

   (b) Although it is not possible to perform the coordinate integrals, the dependence of $Q_{2N}(T, A)$ on A can be obtained by scaling the coordinates: $\boldsymbol{q}'_i = \boldsymbol{q}_i / L$ ($\Rightarrow A' = 1$). Show that this dependence is $Q_{2N}(T, A) \sim A^{2N - N\beta c_0^2/2}$ where $\beta = 1/kT$.

   (c) Calculate the pressure of the gas [$p = (1/\beta)\partial \ln Q_{2N}(T, A)/\partial A$].

2. A system in equilibrium consists of a solid and a vapor (in contact with a thermal bath), both containing one type of atomic particles. Assume that the sublimation energy needed to transform one atom from the solid to the vapor in the limit of infinite atomic mass is known and denote it by ϕ (in other words, ϕ is the depth of the potential energy well). Assume further that the solid can be approximated (Einstein approximation) as a set of completely independent three-dimensional quantum harmonic oscillators (each atom being one oscillator) performing vibrations about their equilibrium position (so that the energy of each atom is $-\phi$ plus the energy due to the harmonic motion) and that the vapor is an ideal gas. Note that the zero-point vibrational energy is not included in the definition of ϕ. Evaluate the vapor pressure as a function of temperature. You may use without proof the partition functions for a one-dimensional harmonic oscillator with frequency ω: $Q_1^{\mathrm{1D}}(V, T) = 1/[2\sinh(\hbar\omega/2kT)]$ and for a free particle of mass m: $Q_1^{\mathrm{gas}}(V, T) = (V/h^3)(2\pi mkT)^{3/2}$.

3. Derive an expression for the quantum mechanical grand canonical partition function for a gas of noniteracting identical particles at temperature T and volume V. Use the occupation numbers formalism. Start from $\mathcal{Z}(T, V, \mu) = \mathrm{Tr}\left(\mathrm{e}^{-\beta(\hat{H} - \mu\hat{N})}\right)$ where $\hat{H}$ is the Hamiltonian of the system, $\hat{N}$ is the number operator, μ is the chemical potential, and $\beta = 1/kT$. Consider the cases when the particles are:

   (a) Bosons (integral spin)

   (b) Fermions (half-integral spin).