PHYS 419: Classical Mechanics Lecture Notes
QUADRATIC AIR RESISTANCE

We will consider motion of a body in air. We will assume that the air resistance can be approximated by the quadratic term only: $\boldsymbol{F}_{\rm drag} = -cv^2\hat{\boldsymbol{v}}$. The motion takes place in Earth gravitational field. We will consider three cases: (i) horizonal motion, (ii) vertical motion, and (iii) general motion.

I. HORIZONTAL MOTION

The Newton equation is

$$m\ddot{x} = -cv_x^2$$

or

$$m\dot{v} = -cv^2$$

where we omit the subscript x for now. This differential equation has a very simple solution

$$\int \frac{dv}{v^2} = -\frac{c}{m}\int dt$$

The integration results in

$$-\frac{1}{v} = -\frac{c}{m}t + C$$

where C is a constant. If $v(0) = v_0$, $C = -1/v_0$ and we have

$$-\frac{1}{v} = -\frac{c}{m}t - \frac{1}{v_0}$$

or

$$\frac{1}{v} = \frac{1}{v_0}\left(\frac{cv_0}{m}t + 1\right).$$

Denoting $m/cv_0 = \tau$, we get

$$v(t) = \frac{v_0}{1 + t/\tau}$$

To get $x(t)$, integrate one more time

$$\int dv = v_0 \int \frac{1}{1 + t/\tau}dt + C$$

so that

$$x(t) = v_0\tau \ln(1 + t/\tau) + C.$$

If $x(0) = 0$, $C = 0$ and we get

$$x(t) = v_0 \tau \ln(1 + t/\tau).$$

This solution has one unexpected feature: since ln is an always increasing function, despite air resistance the object does not have a maximum range, opposite to what we have seen for linear drag. This is an artifact of our restriction to quadratic term only. Although this approximation may work well in some range of velocities, when the velocity becomes very small, the linear component of the air resistance cannot be neglected.

II. VERTICAL MOTION

Let's drop subscript y in v_y, assume $v(0) = 0$ and $y_0 = 0$, and orient the $\hat{\boldsymbol{y}}$ axis downwards. The Newton equation is

$$mg - cv^2 = m\dot{v}$$

We can immediately read from this equation the terminal velocity since when this velocity is reached, $\dot{v} = 0$ and therefore

$$v_{\text{term}} = \sqrt{mg/c}.$$

To solve Newton's equation we write it as

$$\frac{dv}{1 - \frac{c}{mg}v^2} = g dt,$$

use the definition of v_{term}, and integrate

$$\int \frac{dv}{1 - v^2/v_{\text{term}}^2} = g \int dt.$$

We make a substitution $v/v_{\text{term}} = z$, $dz = dv/v_{\text{term}}$ to get

$$v_{\text{term}} \int \frac{dz}{1 - z^2} = gt + C.$$

The integral is elementary

$$\int \frac{dz}{1 - z^2} = \text{arctanh}(z)$$

(arctanh is the inverse function of tanh defined as $\tanh = (e^z - e^{-z})/(e^z + e^{-z})$). Going back to our original variables, we get

$$\text{arctanh}\frac{v}{v_{\text{term}}} = \frac{gt}{v_{\text{term}}} + C$$

or

$$v(t) = v_{\text{term}} \tanh \frac{gt}{v_{\text{term}}} + C'.$$

Since $\tanh(0) = 0$, our initial condition gives $C' = 0$ and the solution becomes

$$v(t) = v_{\text{term}} \tanh \frac{gt}{v_{\text{term}}}.$$

To get $y(t)$, integrate one more time

$$y(t) = \int dv = v_{\text{term}} \int \tanh \frac{gt}{v_{\text{term}}} dt.$$

The integral is

$$\int \tanh(x) dx = \ln(\cosh(x)) + C$$

where $\cosh(x) = (\mathrm{e}^z + \mathrm{e}^{-z})/2$ (the derivation of this formula as well as of some other relations for the hyperbolic functions is the subject of one of the problems, Taylor 2.34, in Assignment 3) so that we get

$$y(t) = \frac{v_{\text{term}}^2}{g} \ln\left(\cosh\left(\frac{gt}{v_{\text{term}}}\right)\right) + C.$$

Since $\cosh(0) = 1$ and $\ln(1) = 0$, our initial condition gives $C = 0$ and the final result is

$$y(t) = \frac{v_{\text{term}}^2}{g} \ln\left(\cosh\left(\frac{gt}{v_{\text{term}}}\right)\right).$$

III. GENERAL (TWO-DIMENSIONAL) MOTION

For a motion involving both components, the Newton's equation is

$$m\ddot{\boldsymbol{r}} = mg\hat{\boldsymbol{y}} - cv^2\hat{\boldsymbol{v}} = mg\hat{\boldsymbol{y}} - cv(v_x\hat{\boldsymbol{x}} + v_y\hat{\boldsymbol{y}}).$$

We can write this vector equation in component form

$$m\ddot{x} = -cvv_x = -c\sqrt{v_x^2 + v_y^2}\; v_x$$

$$m\ddot{y} = mg - c\sqrt{v_x^2 + v_y^2}\; v_y$$

but the two equations are coupled by v. No analytic solution is therefore possible, but the equations can be solved numerically.