I. INTRODUCTION

Classical mechanics is the most axiomatic branch of physics. It is based on only a very few fundamental concepts (quantities, “primitive” terms) that cannot be defined and virtually just one law, i.e., one statement based on experimental observations. All the subsequent development uses only logical reasoning. Thus, classical mechanics is nowadays sometimes considered to be a part of mathematics and indeed some research in this field is conducted at departments of mathematics. However, the outcome of the classical mechanics developments are predictions which can be verified by performing observations and measurements. One spectacular example can be predictions of solar eclipses for virtually any time in the future.

Various textbooks introduce different numbers of fundamental concepts. We will use the minimal possible set, just the concepts of space and time. These two are well-known from everyday experience. We live in a three-dimensional space and have a clear perception of passing of time. We can also easily agree on how to measure these quantities.

We will denote the position of a point in space by a vector \( \mathbf{r} \). If we define an arbitrary Cartesian coordinate system, this vector can be described by a set of three components:

\[
\mathbf{r} = [x, y, z] = x\hat{x} + y\hat{y} + z\hat{z}
\]

where \( \hat{x} \), \( \hat{y} \), and \( \hat{z} \) are unit vectors along the axes of the coordinate system. If the point is moving in the coordinate system, \( \mathbf{r} = \mathbf{r}(t) \). Thus, each component of \( \mathbf{r} \) is a single-variable function, e.g., \( x = x(t) \). The fundamental concepts of time and space allow us to define the velocity \( \mathbf{v} \) and the acceleration \( \mathbf{a} \) of the point in the standard way known from mathematics:

\[
\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}(t)}{dt} = \left[ \frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right]
\]

\[
\mathbf{a} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt}.
\]

II. NEWTON’S LAW

With these definitions, we are ready to formulate Newton’s law. Historically, there were three Newton’s laws developed, and this approach is still followed by many textbooks. We
will relate to these laws later on. To formulate Newton’s law, we have to perform experiments. We will assume that the experiments are performed on two isolated bodies. This approximation can be well realized in practice with modern technology. We can make measurements on balls moving on nearly frictionless surfaces or motions of cars on air cushions. Our laboratory can also be the Universe and we can use telescopes to measure the motions of planets and stars. We measure only the positions of the objects as functions of time. From these data we can calculate the acceleration of each body at each instant of time. The results of these experiments can be summarized in the following law:

**Newton’s Law:** *For two isolated bodies, the ratio of the magnitudes of the acceleration vectors is constant:*

\[
\frac{|a_2|}{|a_1|} = k_{21} = \text{const.,} \quad (1)
\]

*the vectors are parallel\'*

\[a_1 \parallel a_2,\]

*and oriented in the opposite directions.*

\[\begin{align*}
\hat{a}_1 &\equiv \hat{a}_2, \\
\hat{a}_2 &\equiv -\hat{a}_1, \\
\end{align*}\]

where \(|a|\) is the magnitude of vector \(a\).

An alternative way of writing the second and third conditions in Newton’s law is

\[\hat{a}_1 = -\hat{a}_2,\]

where the hat denotes the unit vector in the direction of a given vector, \(\hat{a} = a/|a|\). We can also write the whole Newton’s law as a single equation:

\[a_2 = -k_{21}a_1\]

where \(k_{21}\) is a positive constant (we did not have to say that the constant is positive in Eq. (1) since the ratio of magnitudes is always positive). Indeed, this equation implies that
the acceleration vectors are parallel and oriented in the opposite directions, whereas the equality of the magnitudes of both sides gives Eq. (1).

The ratio of the magnitudes of accelerations is constant for two given bodies, but takes on a different constant value if one of the bodies is replaced by another body. We can write:

\[
\frac{|a_2|}{|a_1|} = k_{21}
\]
\[
\frac{|a_3|}{|a_1|} = k_{31}
\]
\[\ldots\]
\[
\frac{|a_n|}{|a_1|} = k_{n1}.
\]

This ratio allows us to define the concept of mass. The mass is conventionally chosen to be inversely proportional to the acceleration, i.e.,

\[
\frac{|a_2|}{|a_1|} = \frac{1}{m_2} = \frac{m_1}{m_2}
\]
and similarly for the other bodies. Assuming the mass of one of the bodies to be the unit mass, we can measure all the other masses in this way. We therefore have:

**Definition 1** If the mass of body 1 is equal to unity, the mass of body 2 is uniquely defined by the Newton’s Law as

\[
m_2 = \frac{|a_1|}{|a_2|}.
\]

This definition allows us to write Newton’s Law in a concise form:

**Newton’s Law:**

\[
m_1a_1 = -m_2a_2.
\]

This form in turn makes it possible to introduce the concept of force:

**Definition 2** The force is defined as

\[
F = ma.
\]

One should notice that Newton’s Law implicitly states (some textbooks state this explicitly) that the ratio of the magnitudes of accelerations does not depend on how strongly the
two bodies interact, i.e., what are the forces these bodies act on each other with. In fact, as
the bodies move, the distance between them changes all the time and therefore, in general,
the force acting on each body changes as well. Still, the ratio $|a_2|/|a_1|$ remains constant at
all times. The force can even change with time for a given separation between the bodies
(imagine, for example, two bodies connected by a spring which looses its elasticity with
time). This independence of the ratio on the forces allows us to determine masses from
measurements of accelerations.

With the definition of force, we can also write Eq. (2) as $F_1 = -F_2$. Then the ratio of
accelerations is

$$\frac{|a_2|}{|a_1|} = \frac{F_2}{m_2} \frac{F_1}{m_1},$$

where $F_i = |F_i|$, and the forces cancel on the right hand side. This shows that the formulation
given above is true also for the gravitational force, which itself depends on the masses of
interacting bodies.

We can now show the equivalence between our approach and the traditional form of
Newton’s Laws. The Newton’s Law as formulated above is related to the traditional New-
ton’s Third Law: $F_1 = -F_2$. Our definition of the force is the traditional Newton’s Second
Law: $F = ma$. The traditional Newton’s First Law is a trivial theorem resulting from the
definition of the force: if $a = 0$ (or $v =$ const.) $\Leftrightarrow F = 0$.

III. FURTHER DISCUSSION OF NEWTON’S LAW

In its most general form, Newton’s Law specifies that the accelerations, and therefore
the forces, are parallel but not colinear. However, in most cases in classical mechanics the
accelerations and the forces will be colinear, as on the figure. We refer to these as central
forces. The more general form of Newton’s Law is sometimes called the “weak” form. The
form applying to central forces is called the “strong” form. The main example of noncentral forces is the magnetic part of the Lorentz' force $F = q(E + v \times B)$, where $q$ is the charge of the particle, $E$ is the electric field, and $B$ is the magnetic field. If a charged particle is moving near to the pole of a magnet, as shown in the figure, the Lorentz force will be acting out of the page. The force on the pole of the magnet will be—according to Newton’s law—into the page. Thus, the two forces are parallel but not colinear.

The magnetic Lorentz force can also violate Newton’s Law even if the charges move at low velocities. To see this, consider the setup shown below. An exact calculation of the magnetic fields is complicated, but a simple argument viewing the charges as a part of a linear current gives the correct direction of $B$. Then the use of the Lorentz force gives the forces as shown. Clearly, the forces are not colinear. Why nobody seems to be worried about this violation of Newton’s Law? The reason is that at low speeds the electric component of the Lorentz force is so much larger than the magnetic force that the latter is practically unmeasurable. For larger speeds we are going beyond the range of applicability of classical mechanics.

IV. FRAMES OF REFERENCE, SPACE AND TIME, GALILEAN RELATIVITY

The position vectors $r(t)$ have to be measured in some coordinate system. The choice of this system is not arbitrary, and the definition of appropriate systems, called inertial frames of reference, is “operational”:

**Definition 3** An inertial frame of reference is a frame where Newton's law is valid.
Examples of inertial and noninertial frames:

- Braking train—we know from everyday experience that forces act on us despite staying at rest with respect to the car.

- Turntable—clearly not an inertial frame. If you “sit” on a turntable, at rest with respect to the turntable, a force is needed to maintain this position.

- Earth—in most experiments, Earth is assumed to be an inertial frame. However, Earth is in rotational motion, like a turntable. Thus, it is only approximately an inertial system and departures from inertiality can be measured.

- Star-fixed frame—better than Earth, but still an approximation.

The difficulty of classical mechanics to find an absolute inertial system is resolved in relativistic mechanics which requires no inertial frames. For our purposes, the adoption of Earth as an inertial frame of reference will be completely satisfactory for most purposes.

Once we know one inertial reference frame, we can find many other ones. Based on experimental observations, we assume that the space is isotropic, i.e., if frame A is inertial and frame B differs from A only by rotation, Newton’s Law will hold also in the rotated frame (the angular momentum conservation theorem that we will study later is a consequence of the isotropy of space). Other inertial frames are determined by the Galilean (or Newtonian) relativity (or invariance) principle which tells us that

**Theorem 1** If frame A is inertial and frame B moves in a uniform motion with respect to A, then B is an inertial frame as well.

The proof of this theorem is quite obvious. Denote the constant velocity of B relative to A by \( \mathbf{v}_0 \) and the vector connecting the centers of systems at \( t = 0 \) by \( \mathbf{R}_0 \). Assume that the axes of system B are parallel to the respective axes of system A. Consider a point \( \mathbf{r} \) which has coordinates \( \mathbf{r} = (x, y, z) \) in A and \( \mathbf{r} = (x', y', z') \) in B. We have used here the convention that the vector denoting the position of a point is represented by the same symbol in all coordinate systems, but the components of this vector are different in each system. Clearly, \( x'(t) = x(t) - v_{0x} t - X_0 \) and similarly for other components. Consequently, \( \dot{x}' = \dot{x} - v_{0x} \) and \( \ddot{x}' = \ddot{x} \). Since Newton’s Law depends only on second derivatives, this proofs the theorem. A corollary of this theorem is that Newton’s Law is the same in all coordinate systems shifted
with respect to each other. In the proof given above we have assumed that the axes of systems A and B are parallel. If not, we first use the isotropy of space to define a system B’ with the same origin as B, but with axes parallel to A.

Another principle of this type concerns homogeneity of time: experimental results will be the same if an identical experiment is repeated at a later time.

V. CONSERVATION OF MOMENTUM

The momentum $p$ of a body is defined as

$$p = m\dot{r} = mv.$$  

This allows one to write the traditional Newton’s Second Law as

$$F = \dot{p}.$$

For two bodies, we have from the traditional Newton’s Third Law

$$m_1 a_1 = -m_2 a_2$$

or

$$m_1 a_1 + m_2 a_2 = 0.$$  

or

$$\dot{p}_1 + \dot{p}_2 = \frac{d}{dt} (p_1 + p_2) = 0.$$  

However, this means that

$$p_1 + p_2 = \text{const.}$$  

This equation defines the conservation of momentum. As we could see, this is a simple consequence of the traditional Newton’s Third Law and it in fact can be viewed as an alternative formulation of this law.

VI. FORCES

So far we looked at forces as just one way to describe relative motions of two bodies. The origins of the forces are an important subject of past and current research in physics. In our course, we will not need to get into any details of theories of forces. Based on observations, we assume that there are only four fundamental forces in nature:
• gravitational
• electromagnetic
• strong
• weak

Recent astronomical observations indicate that there may exist one more force, sometimes called the dark force, which is responsible for the accelerated expansion of the universe. In our course we will encounter only the first two forces on the list above. The strong force acts between nucleons (protons and neutrons). The weak force is responsible for the $\beta$ decay of nuclei.

A. Gravitational force

The gravitational force acts between any two bodies and for pointwise bodies is given by the Newton’s law of gravity:

$$|F| = G \frac{m_1 m_2}{r^2}$$

where $G$ is a constant and $r$ is the distance between the bodies. The gravitational force is central and always attractive. This is the force we experience everyday. Note that the gravitational attraction between a point near Earth’s surface and the Earth is not between pointwise bodies, but one may show that the force is the same as if the whole mass of the Earth was located in its center. Thus, we may write for a body of mass $m$ interacting with the Earth

$$|F| = G \frac{M m}{r^2} = mg$$

where $M$ is the mass of the Earth. For a body near the surface of the Earth, the distance $r$ is nearly constant so that the quantity $g = GM/r^2$ is nearly constant and it is called gravitational acceleration.
B. Lorentz force

The force due to electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields acting on a body of charge $q$ moving with velocity $\mathbf{v}$ is given by the so-called Lorentz’ force

$$F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

C. Derivative forces

Several other forces which will appear in our course are not new fundamental forces but are macroscopic effects of electromagnetic forces acting on microscopic distances. Among them are:

- normal
- frictional
- retarding

forces.

The normal force is responsible for an object placed on the surface of the table to be at rest despite the action of the gravitational force. The origin of this force are the interactions between atoms forming the table. The frictional force is due to intratomic interactions between two surfaces touching each other. Such interactions are overall attractive, therefore force is needed to move one surface with respect to the other. The retarding forces of fluids (gases or liquids) are due to the same mechanism.