

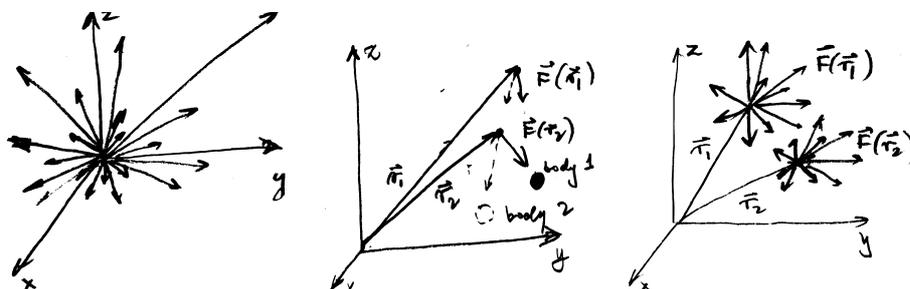
PHYS 419: Classical Mechanics Lecture Notes

I. WORK AND ENERGY

We now want to introduce the concept of energy and derive the mechanical energy conservation theorem. Note the words *derive* and *theorem*. In contrast to most elementary courses on classical mechanics which present this subject as a new physical law, we will show that it is the consequence of Newton's laws. Thus, it is a theorem.

A. Force field

When we have discussed vectors, we stressed that in 3-dimensional space vectors are objects that can be determined by specifying exactly three components (three numbers). These numbers determine the direction, sense, and length of the vector, but not “the origination point” for a vector. Thus, all vectors with the same direction, sense, and length are equivalent. In other words, the origin of a vector is arbitrary and we can for simplicity assume that all vectors originate from the same point in space, for example the point $[0,0,0]$. Then the complete vector space is the set of all possible vectors originating from this point (see the figure).



While this definition is sufficient for many purposes, in physics one often encounters objects that are vectors but are connected with a given point in space. For example, the gravitational force that a body exerts on another body depends on the position in space where we measure this force, see the figure. The force at point \mathbf{r}_1 is different from the force at point \mathbf{r}_2 . If instead of this body (body 1 in the figure), the field is due to another body (body 2), the force vectors are different (broken-line arrows in the figure). By changing the position and mass of the body, one can produce a force of arbitrary direction and magnitude. Thus, at each point of the “position” space there is located a *separate* vector space. We differentiate between the vector spaces located at different \mathbf{r} 's by writing the vectors as $\mathbf{F}(\mathbf{r})$. Now we have six components in each object $\mathbf{F}(\mathbf{r})$, so we can have vectors assigned to different origins. If we chose some value for \mathbf{F} at each \mathbf{r} , we say that a *vector field* has been

determined in space. Note that the vector spaces at different \mathbf{r} are separate vector spaces, so it does not make sense (and there is no need) to talk about the sums like $\mathbf{F}(\mathbf{r}_1) + \mathbf{F}(\mathbf{r}_2)$.

In a general case, the force field can depend on time and it will then be denoted as $\mathbf{F}(\mathbf{r}, t)$. The force may also depend on the velocity of the particle interacting with the field: $\mathbf{F}(\mathbf{r}, \mathbf{v}, t)$.

B. Work performed by a force field

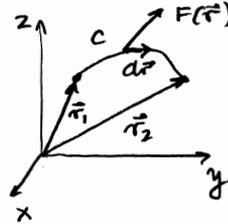
From everyday experience, work is connected with the magnitude of the force acting during the process of performing the work and the magnitude of the displacement of a body. Thus, the following definition should appear to be “natural”. We first define infinitesimal work by the force $\mathbf{F}(\mathbf{r})$ displacing the body from \mathbf{r} to $\mathbf{r} + d\mathbf{r}$ as

$$dW = \mathbf{F} \cdot d\mathbf{r}. \quad (1)$$

Now assume that a body is moved from point \mathbf{r}_1 to point \mathbf{r}_2 along path C , as shown on the figure. Then the work done by the force field \mathbf{F} on the body is

$$W_{12} = \int_{\mathbf{r}_1: C}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \quad (2)$$

In general, the work does depend on the path, so the path has to be specified. The definition of work can be used both for the net force acting on the body (i.e., the sum of all forces) and for each of the particular forces acting on this body.



Note on line integrals:

The line integral appearing in Eq. (2) is defined as follows. We first write the curve C in a parametric form, i.e., the points on the curve are given by three functions

$$\mathbf{r}(t) = [x(t), y(t), z(t)].$$

The beginning and end points correspond then to the values of t equal to t_1 and t_2 , respectively, i.e., $\mathbf{r}_i = \mathbf{r}(t_i)$. Then the definition is

$$\begin{aligned} \int_{\mathbf{r}_1: C}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_{\mathbf{r}_1: C}^{\mathbf{r}_2} [F_x(x, y, z)dx + F_y(x, y, z)dy + F_z(x, y, z)dz] = \\ &= \int_{t_1}^{t_2} [F_x(x(t), y(t), z(t)) x'(t) + F_y(x(t), y(t), z(t)) y'(t) + F_z(x(t), y(t), z(t)) z'(t)] dt \quad (3) \end{aligned}$$

where $x'(t) = dx(t)/dt$. The last integral is just an integral of a single variable, so we know how to compute it. Note that an often made mistake in calculating the line integrals is to use the middle form in Eq. (3) and treat each of the three terms appearing there as one-dimensional integrals (with appropriate assumptions about the values of the other two variables and the limits of integration). While in principle this procedure can lead to the correct answer, in practice it is very easy to make errors in this approach. The only case when such approach can be recommended is when the path of integration consists of linear segments parallel to the axes of the coordinate system.

A useful way of looking at the line integrals is also the Riemann definition. One divides the integration path into a number of segments of length δr . Then the integral is approximated by a sum of (scalar) contributions $\mathbf{F}(\mathbf{r}_\zeta) \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)$, where \mathbf{r}_i are the endpoints of the segments and \mathbf{r}_ζ is a point between \mathbf{r}_i and \mathbf{r}_{i+1} . As the number of segments goes to infinity, the Riemann sum approaches the value of the integral.

C. Kinetic energy

As some forces act on a body, and the net force is nonzero, the body accelerates and the magnitude of the velocity is changing. We know from experience that depending on this magnitude and the mass of the body, the effects of a collision of the body with another object can be quite different. One quantity which depends on velocity and mass is the momentum, a vector quantity. Here we will define a scalar quantity which will reflect this changing state of a body.

Consider again a body of mass m which is moving from point \mathbf{r}_1 to point \mathbf{r}_2 along path C . The body is accelerated. Let us try to relate the work W_{12} to the initial and final velocities of the body, \mathbf{v}_1 and \mathbf{v}_2 . Let $\mathbf{F}(\mathbf{r})$ be the net force acting on the body. We therefore have: $\mathbf{F} = m\mathbf{a}$ and we can write Eq. (2) as

$$\begin{aligned} W_{12} &= \int_{\mathbf{r}_1:C}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = m \int_{\mathbf{r}_1:C}^{\mathbf{r}_2} \mathbf{a} \cdot d\mathbf{r} = m \int_{t_1}^{t_2} \left(\frac{d\mathbf{v}}{dt} \right) \cdot \left(\frac{d\mathbf{r}}{dt} dt \right) \\ &= m \int_{t_1}^{t_2} (\dot{\mathbf{v}} \cdot \mathbf{v}) dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt} dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{dv^2}{dt} dt \\ &= \frac{m}{2} v^2 \Big|_{t_1}^{t_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = T_2 - T_1 \end{aligned} \quad (4)$$

where we have defined

$$T = \frac{mv^2}{2}. \quad (5)$$

This quantity is called the kinetic energy. We have used in the derivation the identity

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \dot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v} \cdot \dot{\mathbf{v}} = 2(\dot{\mathbf{v}} \cdot \mathbf{v}).$$

We have also used the definition of the integral

$$\int \frac{df(x)}{dx} dx = f(x) + C.$$

Thus, the work performed on a particle by the net force is equal to the change of the kinetic energy of this particle:

$$W_{12} = T_2 - T_1 \quad (6)$$

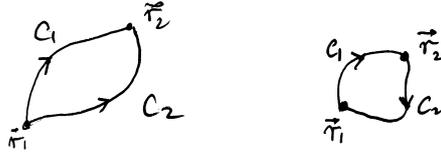
This equation is sometimes called “work–kinetic energy” theorem.

D. Conservative force field

We define the *conservative* force field as a field having the property that the integral (2) is independent of the path C . In other words, the field is conservative if and only if

$$\int_{\mathbf{r}_1:C_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\mathbf{r}_1:C_2}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \quad (7)$$

for any path C_i between the two points (see also the figure below). The reason for using the name “conservative” will be given later.



We can now prove a theorem

Theorem 1 *The force $\mathbf{F}(\mathbf{r})$ is conservative $\Leftrightarrow \oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$.*

Proof (\Rightarrow):

$$\begin{aligned} \oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_{\mathbf{r}_1:C_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_{\mathbf{r}_2:C_2}^{\mathbf{r}_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= \int_{\mathbf{r}_1:C_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} - \int_{\mathbf{r}_1:C_2}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0 \end{aligned} \quad (8)$$

since the field was assumed to be conservative. The paths involved are illustrated on the figure on the right above.

Proof (\Leftarrow):

Consider the figure below. We have two closed path integrals, one over the path $C_1 + C_2$ and another one over $C_1 + C'_2$. Each integral is zero by assumption. Since the integrals have a common part over C_1 , the integral over C_2 must be equal to the integral over C'_2 .



We need one more theorem which introduces the important concept of the potential energy.

Theorem 2 $\oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for any closed path \Leftrightarrow there exists a function $\phi(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$.

Proof (\Rightarrow):

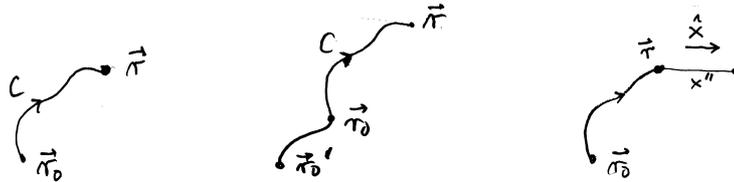
Consider the integral from \mathbf{r}_0 to \mathbf{r} over path C . Due to Theorem 1, this integral is independent of the choice of the path. Thus, for some fixed \mathbf{r}_0 , the integral

$$\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

(there is no C in the symbol of integral to stress the independence of path) gives always the same number for a given \mathbf{r} . Thus, as we change \mathbf{r} , the integral defines a *unique* function of \mathbf{r} :

$$\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = H_{\mathbf{r}_0}(\mathbf{r}). \quad (9)$$

Notice that if we choose another starting point, for example \mathbf{r}'_0 marked on the figure below, the new function $H_{\mathbf{r}'_0}(\mathbf{r})$ differs from $H_{\mathbf{r}_0}(\mathbf{r})$ only by a constant, equal to the value of the integral from \mathbf{r}'_0 to \mathbf{r}_0 .



Now consider the point in space which is shifted with respect to \mathbf{r} by x'' along the $\hat{\mathbf{x}}$ axis (see the figure above on the right). The value of $H_{\mathbf{r}_0}$ at this point is

$$H_{\mathbf{r}_0}(x + x'', y, z) = \int_{\mathbf{r}_0}^{\mathbf{r} + x''\hat{\mathbf{x}}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = H_{\mathbf{r}_0}(x, y, z) + \int_{\mathbf{r}}^{\mathbf{r} + x''\hat{\mathbf{x}}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'. \quad (10)$$

The last integral in the equation above is just a one-dimensional integral since on this path $d\mathbf{r}' = dx'\hat{\mathbf{x}}$. Therefore, we can write

$$\int_{\mathbf{r}}^{\mathbf{r} + x''\hat{\mathbf{x}}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = \int_{\mathbf{r}}^{\mathbf{r} + x''\hat{\mathbf{x}}} F_x(x', y, z) dx' \quad (11)$$

where F_x denotes the x component of \mathbf{F} . In this integral, the values of y' and z' are fixed and equal to y and z , respectively, as utilized in writing the integrand. Thus, the integral written above is equivalent to the standard integral of a single variable $\int g(x')dx'$. Denoting by $G(x)$ the antiderivative of $g(x)$, i.e.,

$$g(x) = \frac{dG(x)}{dx},$$

we can use the definition of the integral:

$$\int_x^{x+x''} g(x')dx' = \int_x^{x+x''} \frac{dG(x')}{dx'} dx' = G(x+x'') - G(x).$$

Now, comparing this to Eqs. (10) and (11), we can write

$$F_x = \frac{\partial H_{\mathbf{r}_0}}{\partial x} \quad (12)$$

Similar relations can be obtained for F_y and F_z by considering appropriate displacements from \mathbf{r} . We therefore can write:

$$\mathbf{F}(\mathbf{r}) = \nabla H_{\mathbf{r}_0}(\mathbf{r}) \quad (13)$$

and the function $\phi(\mathbf{r})$ in the theorem is equal to $H_{\mathbf{r}_0}(\mathbf{r})$ plus an arbitrary constant

$$\phi(\mathbf{r}) = H_{\mathbf{r}_0}(\mathbf{r}) + \text{const.}$$

Proof (\Leftarrow):

If $\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$ then

$$\oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \oint \nabla\phi(\mathbf{r}) \cdot d\mathbf{r} = \oint d\phi = \phi(\mathbf{r}_0) - \phi(\mathbf{r}_0) = 0 \quad (14)$$

where we have used

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \nabla\phi(\mathbf{r}) \cdot d\mathbf{r}.$$

This completes the proof. The intermediate step in (14) can be understood better by writing explicitly the definition of the line integral

$$\oint \nabla\phi(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \nabla\phi(\mathbf{r}) \cdot \dot{\mathbf{r}} dt = \int_{t_1}^{t_2} \left[\frac{\partial\phi}{\partial x} \frac{dx}{dt} + \frac{\partial\phi}{\partial y} \frac{dy}{dt} + \frac{\partial\phi}{\partial z} \frac{dz}{dt} \right] dt = \int_{t_1}^{t_2} \frac{d\phi}{dt} dt.$$

One may remark here that the equation we have proved above

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla\phi(\mathbf{r}) \cdot d\mathbf{r} = \phi(\mathbf{r}_2) - \phi(\mathbf{r}_1)$$

is the three-dimensional equivalent of the familiar formula

$$\int_{x_1}^{x_2} \frac{dG(x)}{dx} dx = G(x_2) - G(x_1).$$

Furthermore, in future work we will often see Eq. (9) written as:

$$\phi(\mathbf{r}) = \phi(\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'.$$

In physics, the function ϕ is traditionally taken with sign reversed, denoted by the letter U , and called the potential energy, so that we have

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}). \quad (15)$$

We can therefore write the relation between work and the potential in one more form

$$U(\mathbf{r}_1) - U(\mathbf{r}_2) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \quad (16)$$

We can now formulate the final form of Theorem 2

Theorem 2a: *For any conservative force field $\mathbf{F}(\mathbf{r})$ there exists a scalar function $U(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$. The function $U(\mathbf{r})$ is called the potential energy.*

As it is obvious from the theorems discussed earlier, the inverse theorem is also true, i.e.,

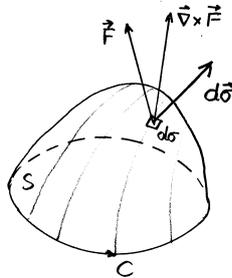
Theorem 2b: *If a force field can be written as $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$, then this force field is conservative.*

E. Curl of \mathbf{F}

There exists another useful relation for checking if a force field is conservative. It utilizes the familiar Stokes theorem

$$\int_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{\sigma} = \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \quad (17)$$

where the first integral is over the surface S and the second integral is over the curve C bounding S . The differential $d\boldsymbol{\sigma}$ is a vector perpendicular to the surface S at a given point and having the magnitude of the surface area element, see the figure. Recall that the curl



of a vector is defined as:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

We can now formulate the following theorem

Theorem 3:

$$\nabla \times \mathbf{F} = 0 \quad \Leftrightarrow \quad \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0 \quad (\Leftrightarrow \quad \mathbf{F} \text{ is conservative})$$

where C is an arbitrary closed contour in some region of space. The proof of this theorem results immediately from the Stokes theorem. In \Rightarrow direction, $\nabla \times \mathbf{F} = 0$ implies that the integral on the left-hand side of Eq. (17) is zero, so the integral on the right-hand side has to be zero as well. In the \Leftarrow direction, we can choose as C any contour on the surface S . Thus, the surface integral similar to that on the left-hand side of Eq. (17), but taken over an arbitrary part of S , must be zero. This is possible only if the integrand is zero. If this is not yet convincing, realize that if the integral on the left-hand side of Eq. (17) (over the complete S) is zero as implied by the assumption, and the integrand is not zero, this means that there are positive and negative contribution to this integral which cancel each other. Then consider the subarea which gives a positive contribution (such that the integrand is positive everywhere). This subarea is bounded by a contour and the line integral over this contour is zero by assumption. Therefore, the surface integral has also to be zero and cannot, in fact, be positive. This implies again that $\nabla \times \mathbf{F} = 0$.

F. Mechanical energy conservation theorem

If a force field is conservative, we can combine Eqs. (2), (15), (16), and (6) together

$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = U(\mathbf{r}_1) - U(\mathbf{r}_2) = U_1 - U_2 = T_2 - T_1$$

which implies

$$T_1 + U_1 = T_2 + U_2.$$

Since the points \mathbf{r}_1 and \mathbf{r}_2 are arbitrary, it follows that

$$T + U = E = \text{const.} \tag{18}$$

where T and U are the kinetic and potential energies, respectively, at some arbitrary point and we have denoted their sum by E . This sum will be called the (total) energy of the system. Equation (18) is the mechanical energy conservation theorem.