Physics 207: Lecture 28

Announcements

- **Final hwk assigned this week, final quiz next week**
- **Review session on Thursday May 19, 2:30 – 4:00pm, Here**

Today's Agenda

- Recap Angular Momentum
- Rotation about a fixed axis
  - ← $L = I\omega$
  - ← Example: Two disks
  - ← Student on rotating stool
- Angular momentum of a freely moving particle
  - ← Bullet hitting stick
  - ← Student throwing ball
- Comment about $\tau = I\alpha$ (not true if I is changing!!)
- Vector considerations of angular momentum
  - ← Bike wheel and rotating stool

1

Angular momentum, *L*

- $\tau_{EXT} = \frac{d\mathbf{L}}{dt}$ where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\tau_{EXT} = \mathbf{r} \times \mathbf{F}_{EXT}$
- In the absence of external torques $\tau_{EXT} = \frac{d\mathbf{L}}{dt} = 0$

Total angular momentum is conserved

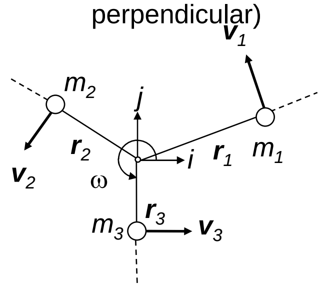
2

Angular momentum of a rigid body about a fixed axis:

- Consider a rigid distribution of point particles rotating in the *x-y* plane around the *z* axis, as shown below. The total angular momentum around the origin is the sum of the angular momenta of each particle:

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \boldsymbol{v}_i = \sum_i m_i r_i v_i \hat{k}$$ (since r_i and v_i are perpendicular)

We see that $\boldsymbol{L}$ is in the *z* direction.



Using $v_i = \omega r_i$, we get

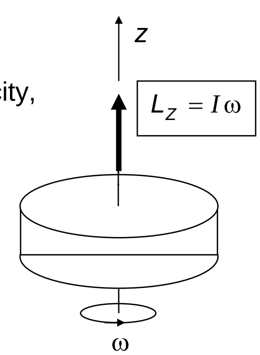
$$L = \sum_i m_i r_i^2 \omega \hat{k}$$

$$\boldsymbol{L} = I\omega$$ Analogue of $\boldsymbol{p} = m\boldsymbol{v}$!!

3

Angular momentum of a rigid body about a fixed axis:

- In general, for an object rotating about a fixed (*z*) axis we can write $L_Z = I\omega$
- The direction of L_Z is given by the right hand rule (same as ω).
- We will omit the *Z* subscript for simplicity, and write $L = I\omega$



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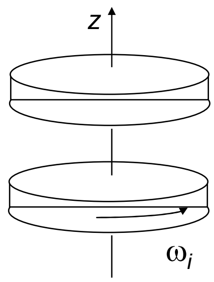
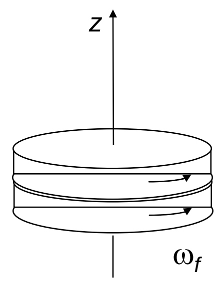
Example: Two Disks

- A disk of mass *M* and radius *R* rotates around the *z* axis with angular velocity ω_i. A second identical disk, initially not rotating, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity ω_f.





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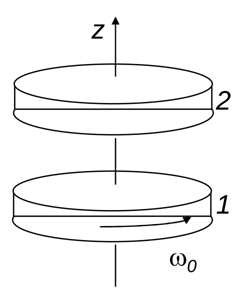
Example: Two Disks

- First realize that there are no external torques acting on the two-disk system.

  ← Angular momentum will be conserved!

- Initially, the total angular momentum is due only to the disk on the bottom:

$$L_i = I_1 \omega_1 = \frac{1}{2} M R^2 \omega_f$$



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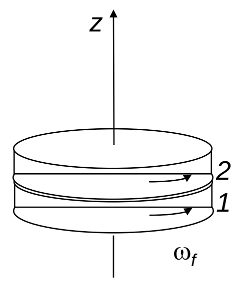
Example: Two Disks

- First realize that there are no external torques acting on the two-disk system.

  ← Angular momentum will be conserved!

- Finally, the total angular momentum is due to both disks spinning:

$$L_f = I_1 \omega_1 + I_2 \omega_2 = MR^2 \omega_f$$



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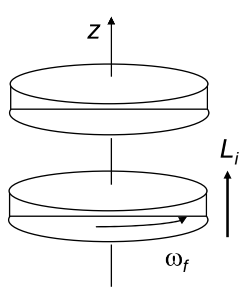
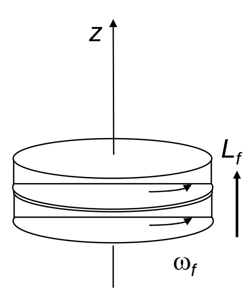
Example: Two Disks

- Since $L_i = L_f$ ⇨ $\frac{1}{2} MR^2 \omega_i = MR^2 \omega_f$

⇨ $\omega_f = \frac{1}{2} \omega_i$

An inelastic collision, since E is not conserved (friction)!





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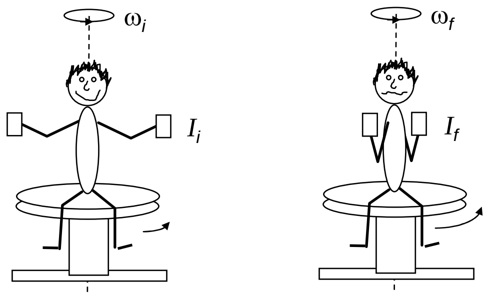
Example: Rotating Table

- A student sits on a rotating stool with his arms extended and a weight in each hand. The total moment of inertia is I_i, and he is rotating with angular speed ω_i. He then pulls his hands in toward his body so that the moment of inertia reduces to I_f. What is his final angular speed ω_f?



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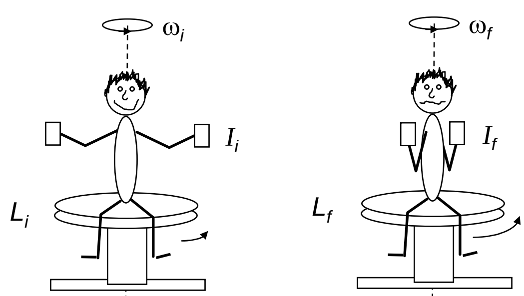
Example: Rotating Table...

- Again, there are no external torques acting on the student-stool system, so angular momentum will be conserved.
  - Initially: $L_i = I_i\omega_i$
  - Finally: $L_f = I_f\omega_f$

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$



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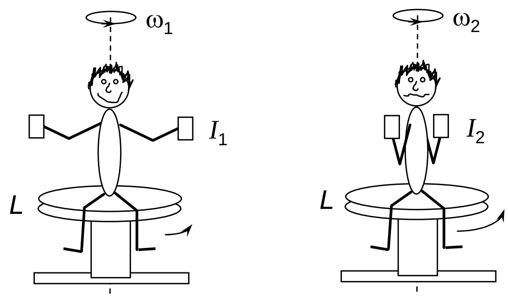
Lecture 28, *Act 1*
Angular Momentum

- A student sits on a freely turning stool and rotates with constant angular velocity ω_1. She pulls her arms in, and due to angular momentum conservation her angular velocity increases to ω_2. In doing this her <u>kinetic energy</u>:

(a) increases (b) decreases (c) stays the same



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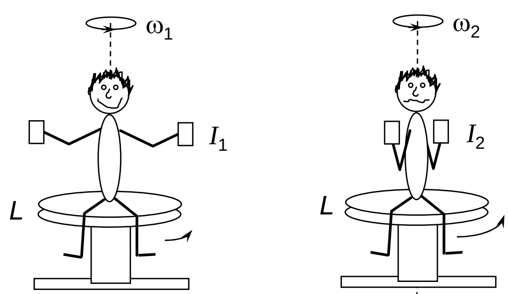
Lecture 28, *Act 1*
Solution

- $K = \frac{1}{2} I\omega^2 = \frac{L^2}{2I}$ (using $L = I\omega$)

- L is conserved:

$I_2 < I_1$ ⇨ $K_2 > K_1$ K increases!



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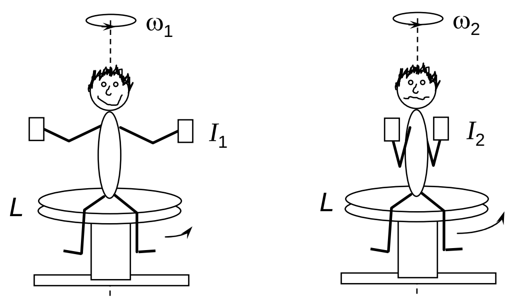
Lecture 28, *Act 1* Solution

- Since the student has to force her arms to move toward her body, she must be doing positive work!
- The work/kinetic energy theorem states that this will increase the kinetic energy of the system!



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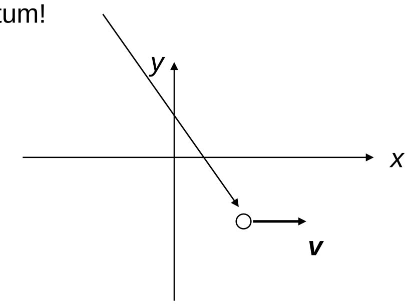
Angular Momentum of a Freely Moving Particle

- We have defined the angular momentum of a particle about the origin as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- This does *not* demand that the particle is moving in a circle!
  - ← We will show that this particle has a constant angular momentum!



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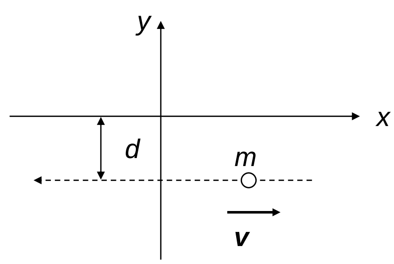
Angular Momentum of a Freely Moving Particle...

- Consider a particle of mass *m* moving with speed *v* along the line $y = -d$. What is its angular momentum as measured from the origin $(0,0)$?



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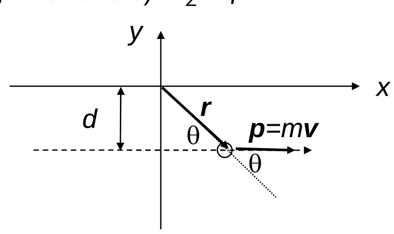
Angular Momentum of a Freely Moving Particle...

- We need to figure out $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- The magnitude of the angular momentum is:

$$|\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = rp \sin\theta = p[r \sin\theta] = pd =$$

$$p \, x \, (distance\ of\ closest\ approach)$$

- Since $\mathbf{r}$ and $\mathbf{p}$ are both in the *x-y* plane, $\mathbf{L}$ will be in the *z* direction (right hand rule): $L_z = pd$



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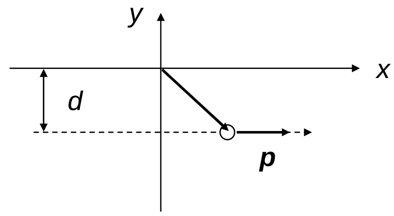
Angular Momentum of a Freely Moving Particle...

- So we see that the direction of ***L*** is along the z axis, and its magnitude is given by $L_Z = pd = mvd$.
- L is clearly conserved since d is constant (the distance of closest approach of the particle to the origin) and p is constant (momentum conservation).



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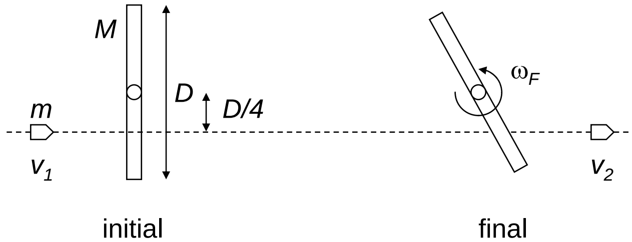
Example: Bullet hitting stick

- A uniform stick of mass M and length D is pivoted at the center. A bullet of mass m is shot through the stick at a point halfway between the pivot and the end. The initial speed of the bullet is v_1, and the final speed is v_2.
  - ➔What is the angular speed ω_F of the stick after the collision? (Ignore gravity)



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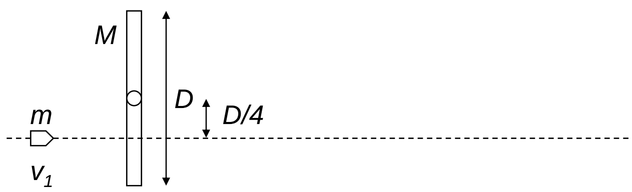
Example: Bullet hitting stick...

- Conserve angular momentum around the pivot (z) axis!
- The total angular momentum before the collision is due only to the bullet (since the stick is not rotating yet).

$$L_i = p\, x\, (distance\ of\ closest\ approach) = mv_1 \frac{D}{4}$$



initial

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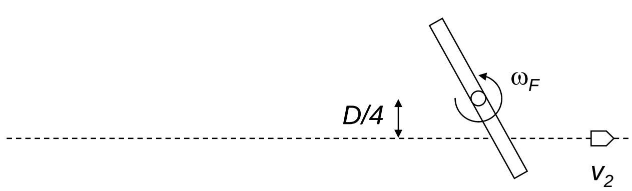
Example: Bullet hitting stick...

- Conserve angular momentum around the pivot (z) axis!
- The total angular momentum after the collision has contributions from both the bullet and the stick.

🡐 $L_f = mv_2 \frac{D}{4} + I\omega_F$ where I is the moment of inertia of the stick about the pivot.



final

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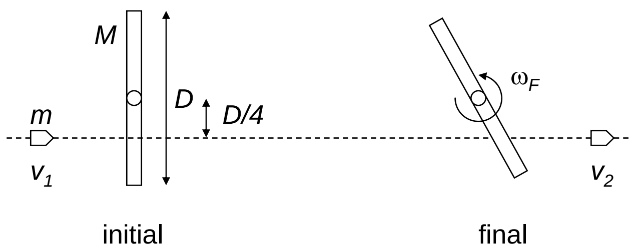
Example: Bullet hitting stick...

- Set $L_i = L_f$ using $I = \frac{1}{12}MD^2$

$$mv_1\frac{D}{4} = mv_2\frac{D}{4} + \frac{1}{12}MD^2\omega_F \Rightarrow \boxed{\omega_F = \frac{3m}{MD}(v_1 - v_2)}$$



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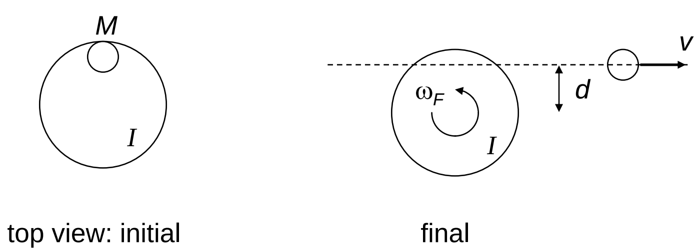
Example: Throwing ball from stool

- A student sits on a stool which is free to rotate. The moment of inertia of the student plus the stool is I. She throws a heavy ball of mass M with speed v such that its velocity vector passes a distance d from the axis of rotation.
  - What is the angular speed ω_F of the student-stool system after she throws the ball?



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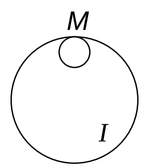
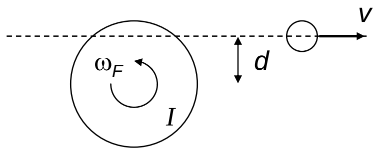
Example: Throwing ball from stool...

- Conserve angular momentum (since there are no external torques acting on the student-stool system):
  - ➔ $L_i = 0$
  - ➔ $L_f = 0 = I\omega_F - Mvd$ ⇨ $\omega_F = \dfrac{Mvd}{I}$





top view: initial final

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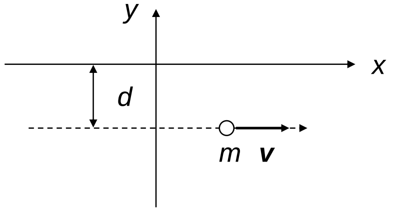
Review...

- A freely moving particle has a definite angular momentum about any axis.
- If no torques are acting on the particle, its angular momentum will be conserved.
- In the example below, the direction of ***L*** is along the *z* axis, and its magnitude is given by $L_z = pd = mvd$.



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Lecture 28, *Act 2*
Angular momentum

- Two different spinning disks have the same angular momentum, but disk 1 has more kinetic energy than disk 2.
  - ← Which one has the biggest moment of inertia?

(a) disk 1 (b) disk 2 (c) not enough info

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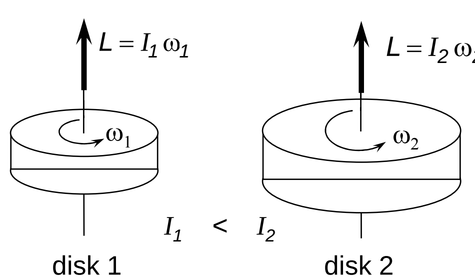
Lecture 28, *Act 2*
Solution

$$K = \frac{1}{2} I \omega^2 = \frac{1}{2I} I^2 \omega^2 = \frac{1}{2I} L^2 \quad \text{(using } L = I\omega \text{)}$$

If they have the same L, the one with the <u>biggest I</u> will have the <u>smallest kinetic energy</u>.



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When does $\tau = I\alpha$ not work ?

- Last time we showed that $\tau_{EXT} = \frac{d\mathbf{L}}{dt}$
- This is the fundamental equation for understanding rotation.
- If we write $L = I\omega$, then

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} + \omega\frac{dI}{dt} = I\alpha + \omega\frac{dI}{dt}$$

$$\boxed{\tau_{EXT} = I\alpha + \omega\frac{dI}{dt}}$$

We can't assume $\tau = I\alpha$ when the moment of inertia is changing!

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When does $\tau = I\alpha$ not work?

$$\boxed{\tau_{EXT} = I\alpha + \omega\frac{dI}{dt}}$$

Now suppose $\tau_{EXT} = 0$:

$$I\alpha + \omega\frac{dI}{dt} = 0 \qquad\qquad \alpha = -\frac{\omega}{I}\frac{dI}{dt}$$

So in this case we can have an α <u>without</u> an external torque!

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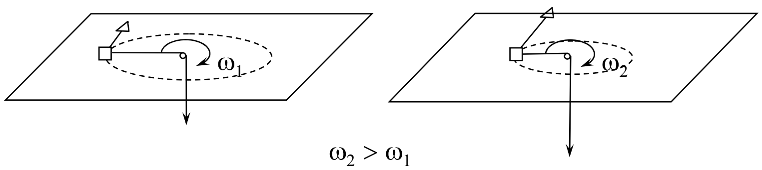
Example...

- A puck in uniform circular motion will experience rotational acceleration if its moment of inertia is changed.

- Changing the radius changes the moment of inertia, but produces no torque since the force of the string is along the radial direction.   (since **r** X **F** = 0)

$I_1 > I_2$



$\omega_2 > \omega_1$

The puck accelerates without external torque!!

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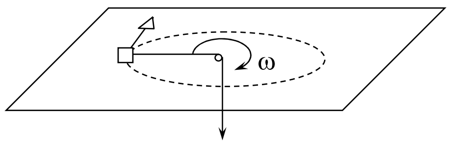
Lecture 28, *Act 3*
Rotations

- A puck slides in a circular path on a horizontal frictionless table. It is held at a constant radius by a string threaded through a frictionless hole at the center of the table. If you pull on the string such that the radius decreases by a factor of *2*, by what factor does the angular velocity of the puck increase?

(a) 2   (b) 4   (c)   8



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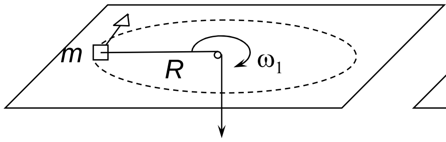
Lecture 28, *Act 3* Solution

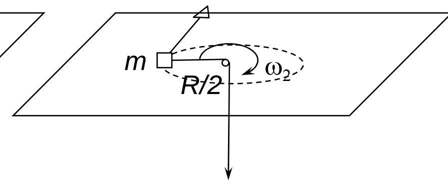
- Since the string is pulled through a hole at the center of rotation, there is no torque: Angular momentum is conserved.

$$L_1 = I_1\omega_1 = mR^2\omega_1 \quad = \quad L_2 = I_2\omega_2 = m\left(\frac{R}{2}\right)^2\omega_2$$

$$mR^2\omega_1 = m\frac{1}{4}R^2\omega_2$$

$$\omega_1 = \frac{1}{4}\omega_2 \Rightarrow \boxed{\omega_2 = 4\omega_1}$$





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Review: Angular Momentum

- $\boxed{\tau_{EXT} = \frac{d\mathbf{L}}{dt}}$ where $\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p}}$ and $\boxed{\tau_{EXT} = \mathbf{r} \times \mathbf{F}_{EXT}}$
- In the absence of external torques $\tau_{EXT} = \frac{d\mathbf{L}}{dt} = 0$

Total angular momentum is conserved

- This is a vector equation.
- Valid for individual components.

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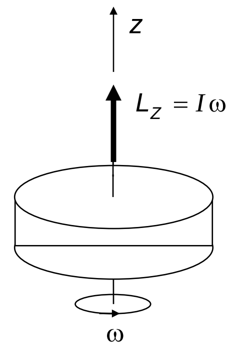
Review...

- In general, for an object rotating about a fixed (z) axis we can write $L_Z = I\omega$
- The direction of L_Z is given by the right hand rule (same as ω).



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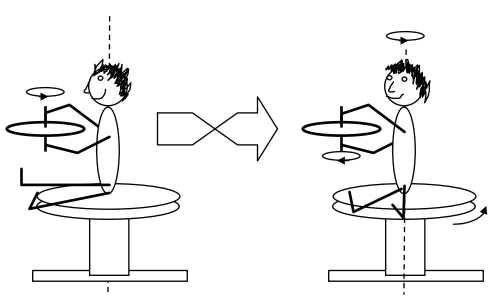
Angular momentum is a vector!
Demo: Turning the bike wheel.

- A student sits on the rotatable stool holding a bicycle wheel that is spinning in the horizontal plane. She flips the rotation axis of the wheel *180º*, and finds that she herself starts to rotate.

  ←What's going on?



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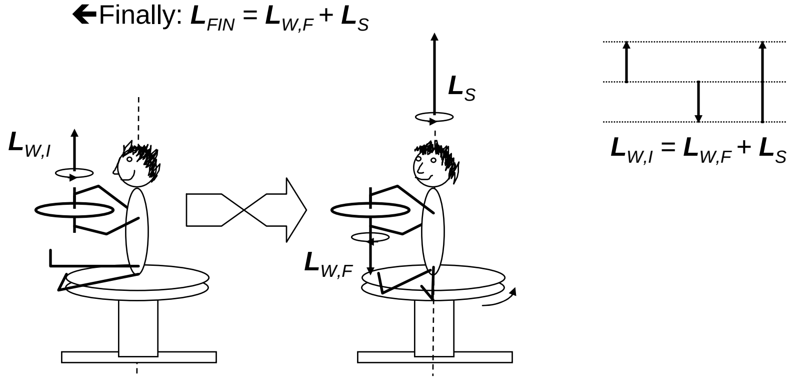
Turning the bike wheel...

- Since there are no external torques acting on the student-stool system, angular momentum is conserved.
  - 🡐Initially: $\mathbf{L}_{INI} = \mathbf{L}_{W,I}$
  - 🡐Finally: $\mathbf{L}_{FIN} = \mathbf{L}_{W,F} + \mathbf{L}_{S}$



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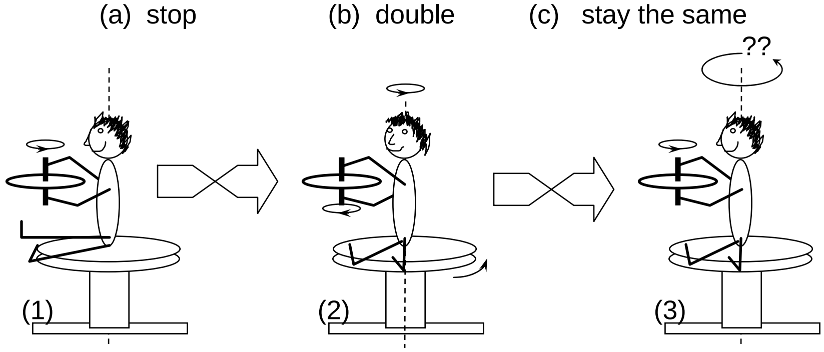
Lecture 28, *Act 4*
Rotations

- A student is initially at rest on a rotatable chair, holding a wheel spinning as shown in (1). He turns it over and starts to rotate (2). If he keeps twisting, turning the wheel over <u>again</u> (3), his rotation will:

(a) stop (b) double (c) stay the same



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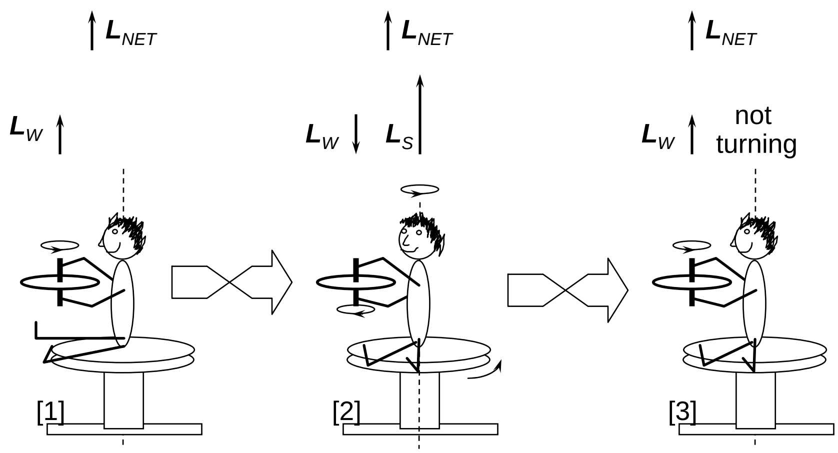
Lecture 28, *Act 4*
Solution



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