PHYS 633 Spring 2008 Project 1 – The Structure of White Dwarf Stars

Structure equations

The equations that describe the structure of stars supported by completely degenerate electron pressure are

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2},$$

and

$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

The pressure is related to the density by

$$p = 6.00\,10^{21} f(x),$$

$$x = 1.01\,10^{-3} \left(\frac{\rho}{\mu_e}\right)^{1/3},$$

and

$$f(x) = x\left(2x^2 - 3\right)\left(x^2 + 1\right)^{1/2} + 3\sinh^{-1} x.$$

The units are SI. Here μ_e is the molecular weight per electron. We can assume $\mu_e = 2$.

The aims of the project are:

1) To find how the stellar radius, *R*, depends on stellar mass, *M*.
2) To find the Chandrasekhar limiting mass, i.e. the upper limit to the mass of a white dwarf.

This requires that the above equations be solved numerically. An efficient way to do this is to start by picking a value for *x* at the center of the star, call it x_c. From this the central values of pressure, p_c, and density, ρ_c, can be found. These can then be used to scale the pressure and density, i.e. let

$$p(r) = p_c Q(r),$$

$$\rho(r) = \rho_c H(r).$$

The structure equations then become

$$\frac{dQ}{dr} = -\frac{\rho_c}{p_c}\frac{GmH}{r^2},$$

$$\frac{dm}{dr} = 4\pi\rho_c r^2 H.$$

To make the other variables order unity, which is advantageous for numerical solution, we need to find convenient length and mass scales. Let

$$r = lz,$$

and

$$m = m_s q(z).$$

The structure equations become

$$\frac{dQ}{dz} = -\frac{\rho_c}{p_c}\frac{Gm_s}{l}\frac{qH}{z^2},$$

and

$$\frac{dq}{dz} = \frac{4\pi l^3 \rho_c}{m_s} z^2 H.$$

For the mass scale, choose

$$m_s = \frac{4\pi}{3} l^3 \rho_c.$$

Now choose *l* to make

$$\frac{\rho_c}{p_c} \frac{G m_s}{l} = 1.$$

This gives

$$l = \sqrt{\frac{3 p_c}{4\pi G \rho_c^2}}.$$

The structure equations are now

$$\frac{dQ}{dz} = -\frac{qH}{z^2},$$

and

$$\frac{dq}{dz} = 3z^2 H,$$

with boundary conditions

$$Q = 1 \text{ at } z = 0,$$

and

$$q = 0 \text{ at } z = 0..$$

From the definitions of *Q* and *H*, we have

$$Q = \frac{f(x)}{f(x_c)},$$

and

$$H = \left(\frac{x}{x_c}\right)^3.$$

Hence it is clearly more convenient to use *x* as a dependent variable instead of *Q*. This gives the final set of equations as

$$\frac{dx}{dz} = -\frac{qH}{z^2} \frac{f(x_c)}{f'(x)},$$

and

$$\frac{dq}{dz} = 3z^2 H,$$

with boundary conditions

$$x = x_c \text{ at } z = 0,$$

and

$$q = 0 \text{ at } z = 0.$$

Note

$$f'(x) = \frac{8x^4}{\sqrt{1+x^2}}.$$

The right hand side of the first equation has a term q/z^2 which needs special consideration at $z = 0$. From the second equation, $q \sim z^3$ near the center and hence $q/z^2 = 0$ at the center.

The equations need to be integrated out from the center to the surface, which is where the density goes to zero, i.e. $x = 0$ at the surface. The location of the surface can be found by interpolation. When the surface value of z is found, call it z_s, the radius and mass are determined from

$$R = lz_s,$$

and

$$M = m_s q(z_s).$$

By considering different values of x_c, a table of R and M values can be found.

1) Make the table of R and M values, and plot R against M using solar radius and solar mass as units.
2) You should find that $R \to 0$ at a finite value of M. This is the Chandrasekhar limiting mass. What is it in units of solar mass?
3) What is the exponent of your mass - radius relation for white dwarfs of low central density? How does this compare with the exponent for a polytrope of polytropic index 3/2?

Solving the differential equations

The equations can easily be solved by using MATLAB or similar software.