PHYS633 Introduction to Stellar Astrophysics Spring 2008

Homework 7: The equilibrium ppI chain

Due in class on Wednesday, April 23rd, 2008

The reactions of the PPI chain are:

$$\begin{aligned} \mathrm{p}+\mathrm{p} &\rightarrow \mathrm{D}+\mathrm{e}^{+}+\nu, \\ \mathrm{D}+\mathrm{p} &\rightarrow {}^{3}\mathrm{He}+\gamma, \\ {}^{3}\mathrm{He}+{}^{3}\mathrm{He} &\rightarrow {}^{4}\mathrm{He}+2\mathrm{p}, \end{aligned}$$

and the corresponding rate equations are

$$\begin{aligned} \frac{dH}{dt} &= -\lambda_{pp}H^{2} - \lambda_{pD}HD + \lambda_{33}\left({}^{3}He\right)^{2}, \\ \frac{dD}{dt} &= -\lambda_{pD}HD + \lambda_{pp}\frac{H^{2}}{2}, \\ \frac{d\,{}^{3}He}{dt} &= -\lambda_{33}\left({}^{3}He\right)^{2} + \lambda_{pD}HD, \\ \frac{d\,{}^{4}He}{dt} &= \lambda_{33}\frac{\left({}^{3}He\right)^{2}}{2}, \end{aligned}$$

where H, D, ${}^{3}He$ and ${}^{4}He$ are the number densities of H, D, ^{3}He and ^{4}He, respectively, and the λ's depend only temperature and density.

1) Show that baryon number is conserved i.e. $H+2D+3\left({}^{3}He\right)+4\left({}^{4}He\right)$ is constant.

2) Assume that deuterium is in equilibrium, so that $dD/dt=0$. Show that the rate equations become

$$\begin{aligned} \frac{dH}{dt} &= -\lambda_{pp}\frac{3H^{2}}{2} + \lambda_{33}\left({}^{3}He\right)^{2}, \\ \frac{d\,{}^{3}He}{dt} &= -\lambda_{33}\left({}^{3}He\right)^{2} + \lambda_{pp}\frac{H^{2}}{2}, \\ \frac{d\,{}^{4}He}{dt} &= \lambda_{33}\frac{\left({}^{3}He\right)^{2}}{2}. \end{aligned}$$

3) Show that when ^3He is also in equilibrium, its abundance is given by

$$\left.\left(\frac{^3He}{H}\right)\right|_{Eq} = \sqrt{\frac{\lambda_{pp}}{2\lambda_{33}}}.$$

Also show that now

$$\frac{dH}{dt} = -\lambda_{pp} H^2.$$

4) From part 3, it follows that the time scale for change in the hydrogen abundance due to the PPI chain is

$$\tau_H = \left(-\frac{d\ln H}{dt}\right)^{-1} = \frac{1}{\lambda_{pp} H}.$$

To estimate the time scale for ^3He to come into equilibrium, suppose the ^3He abundance differs from its equilibrium by a small amount, i.e.

$$^3He = \sqrt{\frac{\lambda_{pp}}{2\lambda_{33}}} H + \varepsilon.$$

Show that the ^3He equation gives

$$\frac{d\varepsilon}{dt} = -\sqrt{2\lambda_{33}\lambda_{pp}} H\varepsilon - \lambda_{33}\varepsilon^2 - \sqrt{\frac{\lambda_{pp}}{2\lambda_{33}}} \frac{dH}{dt}.$$

In general, $\lambda_{33} \gg \lambda_{pp}$ and hence the last term on the RHS is negligible compared to the first. Dropping second order terms, we find that the ^3He time scale is

$$\tau_3 = \left(-\frac{d\ln \varepsilon}{dt}\right)^{-1} = \frac{1}{\sqrt{2\lambda_{33}\lambda_{pp}} H}.$$

Tabulate and graph the ratio of time scales τ_3 / τ_H for temperatures from T_6 = 1 to T_6 = 20. Is the assumption that ^3He in equilibrium always valid?

Use the following expressions for λ_{33} and λ_{pp}:

$$\lambda_{pp} = 3.94\, 10^{-15} \frac{\exp(-3.380 / T_9^{1/3})}{T_9^{2/3}},$$

$$\lambda_{33} = 5.82\, 10^{10} \frac{\exp(-12.276 / T_9^{1/3})}{T_9^{2/3}},$$

where $T_9 = 10^{-3} T_6 = T/10^9$ K.