PHYS633 Introduction to Stellar Astrophysics Spring 2008

Homework 5: Specific heats

Due in class on Monday, March 24th, 2008

1. Consider a mixture of perfect gas and radiation. Verify that

$$C_p = T\left.\frac{\partial s}{\partial T}\right|_p = \frac{\Re}{\mu}\frac{1}{2\beta^2}\left(32 - 24\beta - 3\beta^2\right),$$

and find a similar expression for the C_V, specific heat at constant volume. Compare the ratio of specific heats, $\gamma = C_p / C_V$, to the first adiabatic exponent, Γ_1. Are they equal?

The pressure and internal energy per unit mass are

$$p = \frac{\Re}{\mu}\rho T + \frac{1}{3}aT^4, \tag{1}$$

and

$$u = \frac{3}{2}\frac{\Re}{\mu}T + \frac{aT^4}{\rho}. \tag{2}$$

Since

$$du = Tds + \frac{p}{\rho^2}d\rho, \tag{3}$$

we have

$$C_p = T\left.\frac{\partial s}{\partial T}\right|_p = \left.\frac{\partial u}{\partial T}\right|_p - \frac{p}{\rho^2}\left.\frac{\partial \rho}{\partial T}\right|_p. \tag{4}$$

Since p is kept constant in forming the derivatives with respect to T, we have from equation (1),

$$0 = \frac{\Re}{\mu}Td\rho + \frac{\Re}{\mu}\rho dT + \frac{4}{3}aT^3 dT. \tag{5}$$

Hence

$$\frac{\Re\rho T}{\mu}\frac{d\rho}{\rho} = -\frac{\Re\rho T}{\mu}\frac{dT}{T} - \frac{4}{3}aT^4\frac{dT}{T}, \tag{6}$$

so that

$$\beta\frac{d\rho}{\rho}=-\beta\frac{dT}{T}-4\left(1-\beta\right)\frac{dT}{T}=-\left(4-3\beta\right)\frac{dT}{T}. \tag{7}$$

From equation (2),

$$du=\frac{3}{2}\frac{\Re}{\mu}dT+4\frac{aT^3dT}{\rho}-\frac{aT^4}{\rho^2}d\rho=\frac{p}{\rho}\left[\frac{3}{2}\beta\frac{dT}{T}+12\left(1-\beta\right)\frac{dT}{T}-3\left(1-\beta\right)\frac{d\rho}{\rho}\right]. \tag{8}$$

Using equations (7) and (8), equation (4) gives

$$\begin{aligned}C_p&=\frac{p}{\rho}\left[\frac{3}{2}\beta\frac{1}{T}+12\left(1-\beta\right)\frac{1}{T}-3\left(1-\beta\right)\frac{1}{\rho}\left.\frac{\partial\rho}{\partial T}\right|_p\right]-\frac{p}{\rho^2}\left.\frac{\partial\rho}{\partial T}\right|_p\\&=\frac{p}{\rho T}\left[\frac{3}{2}\beta+12\left(1-\beta\right)-\left(4-3\beta\right)\frac{T}{\rho}\left.\frac{\partial\rho}{\partial T}\right|_p\right]\\&=\frac{\Re}{\mu}\frac{1}{\beta}\left[\frac{3}{2}\beta+12\left(1-\beta\right)+\frac{\left(4-3\beta\right)^2}{\beta}\right]\\&=\frac{\Re}{\mu}\frac{1}{2\beta^2}\left(32-24\beta-3\beta^2\right).\end{aligned} \tag{9}$$

This is the desired result.

Since for unit mass $V=1/\rho$, we have

$$C_V=T\left.\frac{\partial s}{\partial T}\right|_\rho=\left.\frac{\partial u}{\partial T}\right|_\rho=\frac{3}{2}\frac{\Re}{\mu}+4\frac{aT^3}{\rho}=\frac{\Re}{\mu}\left[\frac{3}{2}+4\frac{aT^4}{\frac{\Re\rho T}{\mu}}\right]=\frac{\Re}{\mu}\left[\frac{3}{2}+\frac{12\left(1-\beta\right)}{\beta}\right]=\frac{3\Re}{2\mu}\frac{8-7\beta}{\beta}. \tag{10}$$

The ratio of specific heats is

$$\gamma=\frac{C_p}{C_V}=\frac{1}{\beta}\frac{32-24\beta-3\beta^2}{24-21\beta}, \tag{11}$$

whereas

$$\Gamma_1=\left.\frac{\partial\ln p}{\partial\ln\rho}\right|_s=\frac{32-24\beta-3\beta^2}{24-21\beta}. \tag{12}$$

Hence they are not the same because γ has an additional factor of β^{-1}.

2. Consider the ionization of atomic hydrogen. Verify that

$$C_p = \frac{5}{2}\Re(1+f)\left[1+\frac{1}{5}f(1-f)\left(\frac{5}{2}+\frac{\chi_H}{kT}\right)^2\right].$$

and find a similar expression for the C_V. Find the expression for Γ_1 in similar form to that for ∇_{ad},

$$\nabla_{ad} = \frac{2+f(1-f)\left(\frac{5}{2}+\frac{\chi_H}{kT}\right)}{5+f(1-f)\left(\frac{5}{2}+\frac{\chi_H}{kT}\right)^2}.$$

Again, compare the ratio of specific heats, $\gamma = C_p/C_V$, to the first adiabatic exponent, Γ_1. Are they equal?

The pressure and internal energy per unit mass are

$$p = (1+f)\Re\rho T, \tag{13}$$

and

$$u = \frac{3}{2}(1+f)\frac{kT}{m_H} - (1-f)\frac{\chi_H}{m_H}. \tag{14}$$

Also the Saha equation (in the $G = g_0$ approximation) be written as

$$\frac{f^2}{1-f^2} = \frac{kT}{p}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_H}{kT}\right). \tag{15}$$

In differential form these are

$$\frac{dp}{p} = \frac{df}{1+f} + \frac{d\rho}{\rho} + \frac{dT}{T}, \tag{16}$$

$$du = \frac{3}{2}(1+f)\frac{kT}{m_H}\frac{dT}{T} + \left(\frac{3}{2}\frac{kT}{m_H} + \frac{\chi_H}{m_H}\right)df, \tag{17}$$

and

$$\frac{2}{f(1-f)}\frac{df}{(1+f)} = \left(\frac{5}{2}+\frac{\chi_H}{kT}\right)\frac{dT}{T} - \frac{dp}{p}. \tag{18}$$

If the pressure is kept constant then,

$$\frac{d\rho}{\rho} = -\frac{df}{1+f} - \frac{dT}{T}, \tag{19}$$

and

$$\frac{2}{f(1-f)}\frac{df}{(1+f)}=\left(\frac{5}{2}+\frac{\chi_H}{kT}\right)\frac{dT}{T}. \tag{20}$$

Using equations (17) and (19) in equation (4), we get

$$C_p=\frac{3}{2}(1+f)\frac{k}{m_H}+\left(\frac{3}{2}\frac{kT}{m_H}+\frac{\chi_H}{m_H}\right)\frac{df}{dT}+\frac{p}{\rho}\left(\frac{1}{1+f}\frac{df}{dT}+\frac{1}{T}\right). \tag{21}$$

Now use equation (13) to eliminate the pressure to get

$$C_p=\frac{5}{2}(1+f)\frac{k}{m_H}+\left(\frac{5}{2}\frac{kT}{m_H}+\frac{\chi_H}{m_H}\right)\frac{df}{dT}. \tag{22}$$

Finally use equation (20) to eliminate df/dT to get

$$\begin{aligned}C_p&=\frac{5}{2}(1+f)\frac{k}{m_H}+\left(\frac{5}{2}\frac{kT}{m_H}+\frac{\chi_H}{m_H}\right)\frac{1}{2}\left(\frac{5}{2}+\frac{\chi_H}{kT}\right)\frac{f\left(1-f^2\right)}{T}\\&=\frac{5}{2}\Re(1+f)\left[1+\frac{1}{5}f(1-f)\left(\frac{5}{2}+\frac{\chi_H}{kT}\right)^2\right].\end{aligned} \tag{23}$$

To find C_v, we use the Saha equation in the form

$$\frac{f^2}{1-f}=\frac{m_H}{\rho}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_H}{kT}\right). \tag{24}$$

Keeping the density fixed, this gives

$$\frac{(2-f)}{f(1-f)}df=\left(\frac{3}{2}+\frac{\chi_H}{kT}\right)\frac{dT}{T}. \tag{25}$$

Using equation (17) we have

$$\begin{aligned}C_V&=\left.\frac{\partial u}{\partial T}\right|_\rho=\frac{3}{2}(1+f)\frac{k}{m_H}+\left(\frac{3}{2}\frac{kT}{m_H}+\frac{\chi_H}{m_H}\right)\frac{df}{dT}\\&=\frac{3}{2}(1+f)\frac{k}{m_H}+\left(\frac{3}{2}\frac{kT}{m_H}+\frac{\chi_H}{m_H}\right)\frac{f(1-f)}{(2-f)}\left(\frac{3}{2}+\frac{\chi_H}{kT}\right)\frac{1}{T}\\&=\frac{3}{2}(1+f)\Re\left[1+\frac{2}{3}\frac{f(1-f)}{(1+f)(2-f)}\left(\frac{3}{2}+\frac{\chi_H}{kT}\right)^2\right].\end{aligned} \tag{26}$$

To find Γ_1, we use that for an adiabatic change

$$du = \frac{p}{\rho^2} d\rho. \tag{27}$$

Eliminating the temperature from equations (13) and (14), we have

$$u = \frac{3}{2}\frac{p}{\rho} - (1-f)\frac{\chi_H}{m_H}. \tag{28}$$

Using this in equation (27), we get

$$\frac{\chi_H}{m_H} df = \frac{p}{\rho}\left(\frac{5}{2}\frac{d\rho}{\rho} - \frac{3}{2}\frac{dp}{p}\right). \tag{29}$$

Using the pressure equation, this becomes

$$\frac{\chi_H}{kT} df = (1+f)\left(\frac{5}{2}\frac{d\rho}{\rho} - \frac{3}{2}\frac{dp}{p}\right). \tag{30}$$

Eliminating *dT* from equations (16) and (18), we get

$$\left(\frac{5}{2} + \frac{\chi_H}{kT} + \frac{2}{f(1-f)}\right)\frac{df}{1+f} = \left(\frac{3}{2} + \frac{\chi_H}{kT}\right)\frac{dp}{p} - \left(\frac{5}{2} + \frac{\chi_H}{kT}\right)\frac{d\rho}{\rho}. \tag{31}$$

Eliminating *df* from these last two equations leads to

$$\left(\frac{5}{2} + \frac{\chi_H}{kT} + \frac{2}{f(1-f)}\right)\left(\frac{5}{2}\frac{d\rho}{\rho} - \frac{3}{2}\frac{dp}{p}\right) = \frac{\chi_H}{kT}\left(\frac{3}{2} + \frac{\chi_H}{kT}\right)\frac{dp}{p} - \frac{\chi_H}{kT}\left(\frac{5}{2} + \frac{\chi_H}{kT}\right)\frac{d\rho}{\rho}. \tag{32}$$

Collecting like terms together, we get

$$\left[\frac{3}{2} + \left(\frac{3}{2} + \frac{\chi_H}{kT}\right)^2 + \frac{3}{f(1-f)}\right]\frac{dp}{p} = \left[\left(\frac{5}{2} + \frac{\chi_H}{kT}\right)^2 + \frac{5}{f(1-f)}\right]\frac{d\rho}{\rho}. \tag{33}$$

Hence

$$\Gamma_1 = \frac{5 + f(1-f)\left(\frac{5}{2} + \frac{\chi_H}{kT}\right)^2}{3 + \frac{3}{2}f(1-f) + f(1-f)\left(\frac{3}{2} + \frac{\chi_H}{kT}\right)^2}. \tag{34}$$

We see that $\Gamma_1 = 5/3$ in the limits of completely ionized and completely neutral hydrogen.

The ratio of specific heats is

$$\gamma = \frac{C_p}{C_V} = \frac{5 + f\left(1-f\right)\left(\frac{5}{2} + \frac{\chi_H}{kT}\right)^2}{3 + \frac{2f\left(1-f\right)}{\left(1+f\right)\left(2-f\right)}\left(\frac{3}{2} + \frac{\chi_H}{kT}\right)^2}. \tag{35}$$

Although similar, this is not identical to Γ_1.