PHYS633 Introduction to Stellar Astrophysics Spring 2008

Homework 3: Gravitational Potential Energy

Due in class on Monday, March 10th, 2008

The gravitational potential energy of a star is given by

$$\Omega = -\int_0^M \frac{Gm}{r}\,dm.$$

1) By considering a uniform density model, show explicitly that the gravitational potential φ is not equal to

$$-Gm/r.$$

(The gravitational potential is related to gravitational acceleration by $\mathbf{g} = -\nabla\varphi$.)

For a uniform density model,

$$-\frac{Gm}{r} = -\frac{4\pi G\rho r^2}{3} = -\frac{GM}{R}\frac{r^2}{R^2}.$$

The gravitational acceleration inside the star is

$$\frac{Gm}{r^2} = \frac{4\pi G\rho r}{3} = \frac{d\phi}{dr}.$$

Hence the gravitational potential inside the star

$$\phi = \frac{2\pi G\rho r^2}{3} + C,$$

where C is a constant of integration. The constant of integration can be evaluated from continuity of the potential at the stellar surface. We get

$$-\frac{GM}{R} = \frac{2\pi G\rho R^2}{3} + C.$$

Hence

$$\phi = -\frac{GM}{R} - \frac{2\pi G\rho}{3}\left(R^2 - r^2\right) = -\frac{GM}{R} - \frac{1}{2}\frac{GM}{R}\left(1 - \frac{r^2}{R^2}\right).$$

Clearly the two expressions are not the same.

2) Show that for any spherically symmetric density distribution

$$-\int_0^M \frac{Gm}{r} dm = \frac{1}{2}\int_0^M \varphi dm.$$

(Hint: Use integration by parts.)

Using the hint, we have

$$-\int_0^M \frac{Gm}{r} dm = \left[-\frac{1}{2}\frac{Gm^2}{r}\right]_0^M - \frac{1}{2}\int_0^R \frac{Gm^2}{r^2} dr = -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\int_0^R gmdr$$

$$= -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\int_0^R \frac{d\phi}{dr} mdr = -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\int_0^M \frac{d\phi}{dm} mdm$$

$$= -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\left[\phi m\right]_0^M + \frac{1}{2}\int_0^M \phi dm$$

$$= \frac{1}{2}\int_0^M \phi dm.$$

In the last step, we have used that surface gravitational potential is –GM/R.

3) Show explicitly that the equation in part 2) is satisfied for a uniform density star.

Using the results of part 1), we have for a uniform density star,

$$-\int_0^M \frac{Gm}{r} dm = -\frac{GM}{R}\int_0^M \frac{r^2}{R^2} dm = -\frac{GM}{R}\int_0^R \frac{r^2}{R^2} 4\pi r^2 \rho dr = -\frac{GM}{R}\frac{4\pi\rho R^3}{5} = -\frac{3}{5}\frac{GM^2}{R},$$

and

$$\int_0^M \phi dm = \int_0^M \left[-\frac{GM}{R} - \frac{1}{2}\frac{GM}{R}\left(1 - \frac{r^2}{R^2}\right)\right] dm = -\frac{3}{2}\frac{GM^2}{R} + \frac{1}{2}\frac{GM}{R}\int_0^M \frac{r^2}{R^2} dm$$

$$= -\frac{3}{2}\frac{GM^2}{R} + \frac{3}{10}\frac{GM^2}{R} = -\frac{6}{5}\frac{GM^2}{R}.$$

Hence the equation in part 2) is indeed satisfied for a uniform density star.

4) Explain, on physical grounds, why there is a factor of ½ before the integral on the right hand side of the equation in part 2).

   The factor of ½ arises mathematically to avoid a double counting of pair-wise mass element contributions to the gravitationally binding energy. From a physical view point, the factor of ½ arises because the potential well gets shallower as mass is removed from the star.