PHYS 633 Introduction to Stellar Astrophysics Spring 2008

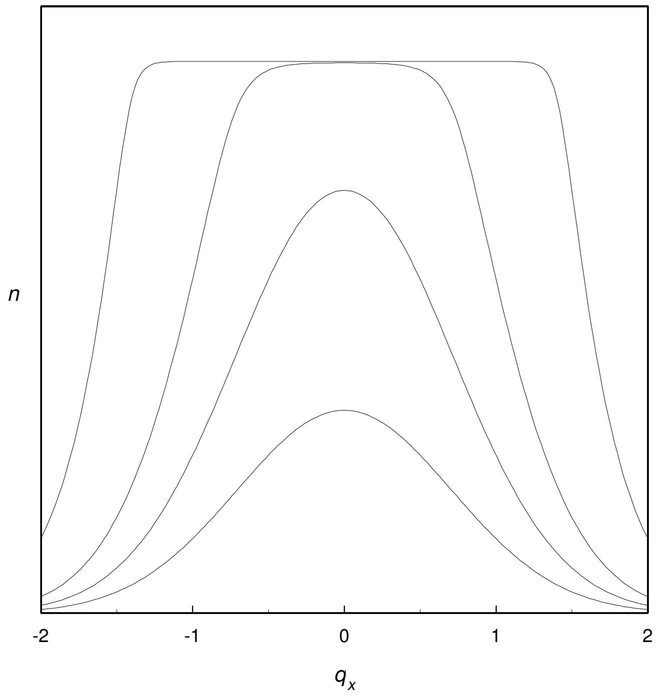
The degenerate electron gas

8.1 Introduction

In dense material such as is found in white dwarfs and the cores of red giant star, the electrons are sufficiently close together that the quantum nature of phase space must be taken into account. This leads to what is called degenerate electron pressure.

Since electrons are spin ½ particles they obey Fermi-Dirac statistics and Pauli's exclusion principle. For free electrons, Pauli's exclusion principle gives that no more than 2 electrons can occupy a volume of h^3 in phase space. Here h is Planck's constant.

To visualize what this means, restrict to one dimension in momentum space and consider unit volume of position space. The graph below shows the distribution in momentum for finite temperature and 4 values of electron density.



At low density, the distribution is Maxwellian. The Pauli Exclusion Principle sets a ceiling on density in momentum space. At high density all states with momentum less than a threshold are occupied and very few electrons have momentum greater then this threshold.

We say that the electrons are *degenerate* if the Pauli Exclusion Principle significantly modifies the momentum distribution from Maxwellian.

8.2 Complete electron degeneracy

This occurs at $T = 0$. In this case all states with momentum less than the threshold are filled and states with momentum greater than the threshold are empty. Let the threshold momentum be q_f. This is called the Fermi momentum.

The electron number density is

$$n_e = \frac{2}{h^3}\int_0^{q_f} 4\pi q^2 dq = \frac{2}{h^3}\frac{4\pi}{3}{q_f}^3. \tag{8.2.1}$$

To calculate the electron pressure we consider the momentum flux in the x-direction

$$p_e = \iiint v_x q_x \frac{2}{h^3} dq_x dq_y dq_z. \tag{8.2.2}$$

By isotropy

$$3p_e = \iiint \left(v_x q_x + v_y q_y + v_z q_z\right)\frac{2}{h^3} dq_x dq_y dq_z = \int_0^{q_f} vq \frac{2}{h^3} 4\pi q^2 dq. \tag{8.2.3}$$

The electrons might be relativistic and hence we must include the Lorentz factor

$$q = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{8.2.4}$$

$$\frac{v}{c} = \frac{1}{\sqrt{1+\left(\frac{q}{mc}\right)^2}}\frac{q}{mc}, \tag{8.2.5}$$

where m is the electron mass.

Hence

$$p_e = \frac{8\pi}{3h^3}\int_0^{q_f} vq^3 dq = \frac{8\pi}{3mh^3}\int_0^{q_f}\frac{q^4 dq}{\sqrt{1+\left(\frac{q}{mc}\right)^2}}. \tag{8.2.6}$$

The integral is evaluated by making the substitution $q = mc\sinh\theta,$ which gives

$$p_e = \frac{\pi m^4 c^5}{3h^3} f\left(x\right), \tag{8.2.7}$$

where

$$f\left(x\right)=x\left(2x^2-3\right)\left(x^2+1\right)^{1/2}+3\sinh^{-1}x, \tag{8.2.8}$$

and

$$x=\frac{q_f}{mc}, \tag{8.2.9}$$

is a dimensionless Fermi momentum or “relativity parameter”.

The electron number density is given in terms of the matter density by

$$n_e=\frac{\rho}{m_u\mu_e}, \tag{8.2.10}$$

where μ_e is the mean molecular weight per electron. This allows us to write q_f and x in terms of the matter density. We find

$$x=\left(\frac{3}{8\pi}\right)^{1/3}\frac{h}{mc}\left(\frac{\rho}{m_u\mu_e}\right)^{1/3}. \tag{8.2.11}$$

In SI units

$$p_e=6.00\ 10^{21}f\left(x\right)\ \mathrm{N\ m^{-2}}, \tag{8.2.12}$$

and

$$x=1.01\ 10^{-3}\left(\frac{\rho}{\mu_e}\right)^{1/3}, \tag{8.2.13}$$

The energy density (energy per unit volume) of the completely degenerate electrons (including their rest mass energy) is

$$\varepsilon=\frac{2}{h^3}\int_0^{q_f}mc^2\left(1+\frac{q^2}{m^2c^2}\right)^{1/2}4\pi q^2dq=\frac{2m^3c^3}{h^3}mc^2\int_0^x\left(1+u^2\right)^{1/2}4\pi u^2du. \tag{8.2.14}$$

Making the same substitution as for the pressure, we get

$$\varepsilon=\pi\left(\frac{mc}{h}\right)^3mc^2\left[x\left(1+x^2\right)^{1/2}\left(1+2x^2\right)-\sinh^{-1}x-\frac{8}{3}x^3\right]+\pi\left(\frac{mc}{h}\right)^3mc^2\frac{8}{3}x^3, \tag{8.2.15}$$

where the last term is the rest mass energy density.

8.3 Limiting forms

The limits $x \ll 1,$ and $x \gg 1$ correspond to non-relativistic and relativistic electrons respectively. The relation between pressure and density in these limits is most easily obtained by considering equation (8.2.6) in these limits.

For non-relativistic electrons, $q \ll mc,$ and so

$$p_e \approx \frac{8\pi}{3mh^3}\int_0^{q_f} q^4 dq = \frac{8\pi}{15mh^3} q_f{}^5 = \frac{8\pi m^4 c^5}{15h^3} x^5 = \frac{h^2}{5m}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho}{m_u \mu_e}\right)^{5/3} = 1.00\ 10^7 \left(\frac{\rho}{\mu_e}\right)^{5/3}\ \mathrm{N\ m^{-2}}. \quad (8.3.1)$$

For highly relativistic electrons

$$p_e = \frac{8\pi c}{3h^3}\int_0^{q_f} q^3 dq = \frac{2\pi c}{3h^3} q_f{}^4 = \frac{2\pi m^4 c^5}{3h^3} x^4 = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc \left(\frac{\rho}{m_u \mu_e}\right)^{4/3} = 1.24\ 10^{10} \left(\frac{\rho}{\mu_e}\right)^{4/3}\ \mathrm{N\ m^{-2}}. \quad (8.3.2)$$

We can estimate the density, $\rho_{nr\_r},$ at which the transition from non-relativistic to relativistic occurs by comparing these two expressions for the electron pressure. We find

$$\frac{\rho_{nr\_r}}{\mu_e} \approx 10^9\ \mathrm{kg\ m^{-3}}. \quad (8.3.3)$$

The limiting forms of the internal energy per unit volume are

$$\varepsilon = \frac{8\pi m^3 c^3}{h^3} mc^2 \left(\frac{x^2}{3} + \frac{x^5}{10} + \cdots\right) \quad (8.3.4)$$

for $x \ll 1,$ and

$$\varepsilon = \frac{2\pi m^3 c^3}{h^3} mc^2 \left(x^4 + x^2 + \cdots\right), \quad (8.3.5)$$

for $x \gg 1.$ Note these expressions include the rest mass energy.

8.4 The contribution from nuclei at zero temperature

The contribution to the pressure from completely degenerate nuclei is negligible. To see why suppose for simplicity that the nuclei are protons of mass *M*. By charge neutrality, the proton number density is the same as the electron number density and hence the Fermi momentum is the same. The dimensional Fermi parameter for the protons is smaller than that for the electrons by a factor *m*/*M*. Hence the nuclei are much less relativistic than the electrons. From equation (8.3.1), the pressure from the nuclei is

$$p_{nuc} \approx \frac{8\pi}{15Mh^3} q_0{}^5 = \frac{m}{M}\frac{8\pi m^4 c^5}{15h^3} x^5 = 5.51\ 10^3 \left(\frac{\rho}{\mu_e}\right)^{5/3}\ \mathrm{N\ m^{-2}}. \quad (8.4.1)$$

This is small compared to the pressure from non-relativistic electrons and becomes comparable to that from relativistic electrons when

$$\frac{\rho}{\mu_e} \approx 10^{19} \text{ kg m}^{-3}.$$

This is greater than nuclear densities and hence for cold white dwarfs, the contribution to the pressure from completely degenerate nuclei is negligible.

8.5 Transition from non-degeneracy to degeneracy

At finite temperature and low enough density, the electron momentum distribution will be Maxwellian and the electron pressure can be found from the perfect gas law

$$p_e = n_e kT = \frac{\rho}{m_u \mu_e} kT. \tag{8.5.1}$$

We can find the density, ρ_{nd_d}, at which the electrons start to become degenerate by comparing the expressions for electron pressure in equations (8.3.1) and (8.5.1):

$$\frac{h^2}{5m}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho_{nd\_d}}{m_u \mu_e}\right)^{5/3} = \frac{\rho_{nd\_d}}{m_u \mu_e} kT. \tag{8.5.2}$$

This gives

$$\frac{\rho_{nd\_d}}{\mu_e} = \frac{8\pi m_u}{3}\left(5\frac{mkT}{h^2}\right)^{3/2} = 2.5\,10^{-5} T^{3/2} \text{ kg m}^{-3}. \tag{8.5.3}$$

At the solar center, $T \approx 1.5\,10^{7}$ K, and so $\rho_{nd\_d} \sim 2\,10^{6} \text{ kg m}^{-3}$, which is greater than the solar central density by a factor of about 20. Hence the electrons in the Sun are non-degenerate.

8.6 Effects of degeneracy on the adiabatic gradient and the first adiabatic exponent

To see how electron degeneracy affects that adiabatic gradient and the first adiabatic exponent, first consider a situation in which the electrons are non-relativistic but sufficiently degenerate that equation (8.3.1) gives a good approximation to the electron pressure. Adding the contribution from the non-degenerate ions, the total pressure is

$$p = \frac{h^2}{5m}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho}{m_u \mu_e}\right)^{5/3} + \frac{\rho}{m_u \mu_{ion}} kT, \tag{8.6.1}$$

where μ_{ion} is the mean molecular weight per ion. The molecular weights are related by

$$\sum_k \frac{X_k}{A_k}\left(1+Z_k\right)=\frac{1}{\mu}=\frac{1}{\mu_{ion}}+\frac{1}{\mu_e}=\sum_k \frac{X_k}{A_k}+\sum_k \frac{X_k Z_k}{A_k}, \tag{8.6.2}$$

where Z_k is the number of electrons freed from atoms of species k.

The internal energy per unit mass (excluding rest mass energy) is

$$u=\frac{3}{2}\frac{h^2}{5m\rho}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho}{m_u\mu_e}\right)^{5/3}+\frac{3}{2}\frac{kT}{m_u\mu_{ion}}. \tag{8.6.3}$$

Comparing equations (8.6.1) and (8.6.3), we see that

$$u=\frac{3}{2}\frac{p}{\rho}, \tag{8.6.4}$$

just as for the perfect gas. Since for adiabatic change

$$du=\frac{p}{\rho}\frac{d\rho}{\rho}, \tag{8.6.5}$$

we have

$$du=\frac{3}{2}\frac{dp}{\rho}-\frac{3}{2}\frac{p}{\rho}\frac{d\rho}{\rho}=\frac{p}{\rho}\frac{d\rho}{\rho}. \tag{8.6.6}$$

We immediately see that

$$\Gamma_1=\frac{5}{3}. \tag{8.6.7}$$

Also, on using the expressions for u and p in equation (8.6.5), we get

$$\frac{h^2}{5m\rho}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho}{m_u\mu_e}\right)^{5/3}\frac{d\rho}{\rho}+\frac{3}{2}\frac{k}{m_u\mu_{ion}}dT=\frac{h^2}{5m\rho}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{\rho}{m_u\mu_e}\right)^{5/3}\frac{d\rho}{\rho}+\frac{k}{m_u\mu_{ion}}T\frac{d\rho}{\rho}. \tag{8.6.8}$$

The first terms on each side cancel, and we get

$$\frac{3}{2}\frac{dT}{T}=\frac{d\rho}{\rho}. \tag{8.6.9}$$

Since

$$\nabla_{ad}=\left.\frac{\partial \ln T}{\partial \ln p}\right|_s=\left.\frac{\partial \ln T}{\partial \ln \rho}\right|_s\Big/\left.\frac{\partial \ln p}{\partial \ln \rho}\right|_s=\frac{1}{\Gamma_1}\left.\frac{\partial \ln T}{\partial \ln \rho}\right|_s, \tag{8.6.10}$$

we find that $\nabla_{ad} = 2/5$, again the same as for a perfect gas.

Now consider a situation in which the electrons are highly relativistically degenerate. The total pressure and internal energy per unit mass are

$$p = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} + \frac{\rho}{m_u\mu_{ion}}kT, \tag{8.6.11}$$

and

$$u = \frac{3}{4}\left(\frac{3}{8\pi}\right)^{1/3} \frac{ch}{\rho}\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} + \frac{3}{2}\frac{kT}{m_u\mu_{ion}}. \tag{8.6.12}$$

For an adiabatic change, we get

$$\frac{1}{3}\frac{3}{4}\left(\frac{3}{8\pi}\right)^{1/3} \frac{ch}{\rho}\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} \frac{d\rho}{\rho} + \frac{3}{2}\frac{kT}{m_u\mu_{ion}}\frac{dT}{T} = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{\rho}\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} \frac{d\rho}{\rho} + \frac{kT}{m_u\mu_{ion}}\frac{d\rho}{\rho}. \tag{8.6.13}$$

Again, the first terms on each side cancel, and so

$$\frac{3}{2}\frac{dT}{T} = \frac{d\rho}{\rho}. \tag{8.6.14}$$

From equation (8.6.11),

$$dp = \frac{1}{3}\left(\frac{3}{8\pi}\right)^{1/3} hc\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} \frac{d\rho}{\rho} + \frac{\rho kT}{m_u\mu_{ion}}\frac{d\rho}{\rho} + \frac{\rho kT}{m_u\mu_{ion}}\frac{dT}{T}, \tag{8.6.15}$$

so that for an adiabatic change

$$\frac{dp}{p} = \frac{\frac{1}{2}\left(\frac{3}{8\pi}\right)^{1/3} hc\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} + \frac{5}{2}\frac{\rho kT}{m_u\mu_{ion}}}{\frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc\left(\frac{\rho}{m_u\mu_e}\right)^{4/3} + \frac{\rho kT}{m_u\mu_{ion}}}\frac{dT}{T} = \frac{2p_e + \frac{5}{2}p_{ion}}{p_e + p_{ion}}\frac{dT}{T}. \tag{8.6.16}$$

Hence for highly relativistic degenerate electrons,

$$\nabla_{ad} = \frac{p_e + p_{ion}}{2p_e + \frac{5}{2}p_{ion}}, \tag{8.6.17}$$

and

$$\Gamma_1 = \frac{4p_e + 5p_{ion}}{3p_e + 3p_{ion}}. \tag{8.6.18}$$

If the electron pressure dominates (which is the usual case when the electrons are relativistically degenerate), then

$$\nabla_{ad} = \frac{1}{2}, \tag{8.6.19}$$

and

$$\Gamma_1 = \frac{4}{3}. \tag{8.6.20}$$