

PHYS 633 Introduction to Stellar Astrophysics

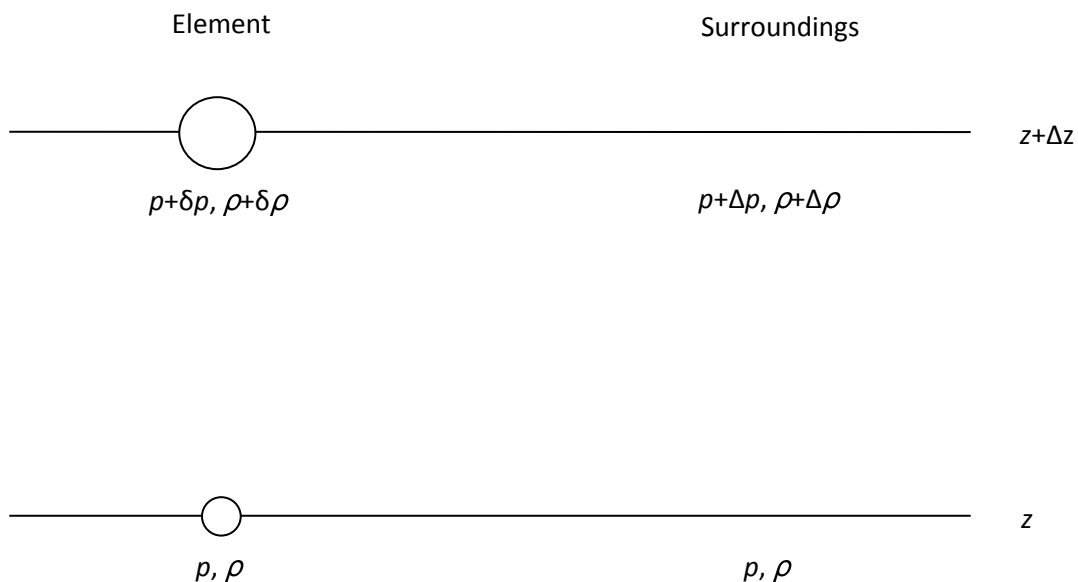
Spring 2008

Convective energy transport

Introduction

Convection is due to an instability that arises when the temperature gradient exceeds a critical value. There is as yet no simple but accurate way of including convective energy transport in models of stars. A phenomenological model of convection called *mixing length theory* is most often used.

The Schwarzschild criterion for convective instability



To derive the conditions under which convective instability will occur, we consider a region of the star that is in radiative equilibrium. Suppose a small volume element at height z is moved to height $z + \Delta z$ slowly enough that it remains in pressure balance with its surroundings but quickly enough that it does not lose energy to its surroundings. This means that the entropy of the element does not change, i.e. it is moved adiabatically.

The change in pressure in the element and the surroundings is

$$\delta p = \Delta p = \frac{dp}{dz} \Delta z.$$

Because the element is moved adiabatically, the change in the pressure inside the element and its density are related by

$$\frac{\delta\rho}{\rho} = \Gamma_1 \frac{\delta p}{\rho},$$

where Γ_1 is the first adiabatic exponent (this expression is actually the definition of the first adiabatic exponent). Γ_1 is, in principle, a known function of density, temperature and composition.

The change in density of the element is then

$$\delta\rho = \frac{\rho}{\Gamma_1} \frac{\delta p}{\rho} = \frac{\rho}{\Gamma_1} \frac{\Delta p}{\rho} = \frac{\rho}{\Gamma_1} \frac{d \ln \rho}{dz} \Delta z.$$

The change in density of the surroundings is

$$\Delta\rho = \frac{d\rho}{dz} \Delta z.$$

If the element is lighter than its surroundings at height $z + \Delta z$ it will continue to move up. If this is the case, the layer at height z is convectively unstable. The condition for instability is

$$-\delta\rho > -\Delta\rho.$$

Using the above expressions for the changes in density, this becomes

$$-\frac{\rho}{\Gamma_1} \frac{d \ln \rho}{dz} \Delta z > -\frac{d\rho}{dz} \Delta z.$$

(Note that both sides of the above inequality are positive in radiative layers). Hence the layer is convectively unstable if

$$\frac{d \ln \rho}{d \ln \rho} > \Gamma_1$$

where

$$\frac{d \ln \rho}{d \ln \rho} = \frac{-\frac{d \ln \rho}{dz}}{-\frac{d \ln \rho}{dz}}.$$

This is the *Schwarzschild criterion* for convection. Because the radiative flux is related to the temperature gradient, it is useful to re-write the Schwarzschild condition in terms of the temperature gradient. We can use the property of pressure balance to transform to the temperature gradients:

$$\delta p = \left. \frac{\partial p}{\partial \rho} \right|_{\tau} \delta \rho + \left. \frac{\partial p}{\partial \tau} \right|_{\rho} \delta \tau = \Delta p = \left. \frac{\partial p}{\partial \rho} \right|_{\tau} \Delta \rho + \left. \frac{\partial p}{\partial \tau} \right|_{\rho} \Delta \tau.$$

Re-arranging, we find

$$\left. \frac{\partial \rho}{\partial \rho} \right|_T (\delta \rho - \Delta \rho) = - \left. \frac{\partial \rho}{\partial T} \right|_\rho (\delta T - \Delta T).$$

In most circumstances $\left. \frac{\partial \rho}{\partial \rho} \right|_T$, $\left. \frac{\partial \rho}{\partial T} \right|_\rho$ are positive. Hence the instability condition

$$\delta \rho - \Delta \rho < 0,$$

becomes

$$\delta T - \Delta T > 0,$$

i.e. the element must be hotter than its surroundings to continue rising. Since $\delta \rho = \Delta \rho < 0$, this gives

$$\frac{\delta T}{\delta \rho} - \frac{\Delta T}{\Delta \rho} < 0.$$

Since the element is moved adiabatically,

$$\frac{\delta T}{\delta \rho} = \left. \frac{\partial \ln T}{\partial \ln \rho} \right|_s \frac{T}{\rho} = \nabla_{ad} \frac{T}{\rho},$$

where ∇_{ad} is the *adiabatic gradient*.

Also since

$$\Delta \rho = \frac{d\rho}{dz} \Delta z, \Delta T = \frac{dT}{dz} \Delta z,$$

the Schwarzschild criterion for convective instability can be written as

$$\nabla \equiv \frac{d \ln T}{d \ln \rho} > \nabla_{ad}$$

where ∇ is the structural gradient.

Note that in deriving the form of the Schwarzschild criterion in terms of temperature gradients, we have assumed that the star is chemically homogeneous. There is some debate as to the correct criterion for convective instability in the presence of composition gradients. However when radiative energy transfer is included this condition remains correct even in the presence of composition gradients.

Including convective energy transport in stellar models

If a region of a star is convective, then there is an additional 'channel' for transporting energy, which adds to the radiative energy flux. If all the energy were transported radiatively, the temperature gradient would be such that

$$\frac{d \ln T}{d \ln p} = \frac{\frac{d \ln T}{dr}}{\frac{d \ln p}{dr}} = \frac{\frac{3\kappa p L}{16\pi a c r^2 T^4}}{\frac{G m \rho}{\rho r^2}} = \frac{3\kappa p L}{16\pi a c G m T^4} \equiv \nabla_{rad}.$$

This expression is called the *radiative gradient*. Because it carries part of the energy flux, convection acts to reduce the structural gradient below the radiative gradient. Hence in convective regions, the ordering of the three gradients is

$$\nabla_{rad} \geq \nabla \geq \nabla_{ad}.$$

If convection is very efficient, then ∇ becomes very nearly equal to ∇_{ad} . Main sequence stars more massive than the Sun have convective cores. In these convective cores, the density is high so that thermal content of the convective elements is large and convection is efficient. If convection is inefficient, then very little energy is transported by convection and so $\nabla \approx \nabla_{rad}$.

To model intermediate conditions, which occur for example in the outer layers of the Sun, a simple phenomenological mixing length theory is used. The mixing length can be thought of as the characteristic size of the convective cells or the average distance a convective element moves vertically before dissolving into the background and depositing its heat.

Let the mixing length be l . After moving a mixing length in the vertical direction, the convective element is hotter than its surroundings by

$$\delta T - \Delta T = l T \left(\left. \frac{d \ln T}{dz} \right|_{ad} - \frac{d \ln T}{dz} \right) = l T \left(\left. \frac{d \ln T}{d \ln p} \right|_{ad} - \frac{d \ln T}{d \ln p} \right) \frac{d \ln p}{dz} = l T (\nabla - \nabla_{ad}) \frac{\rho g}{p}.$$

The convective energy flux is

$$F_{conv} \approx v \rho C_p (\delta T - \Delta T),$$

where v is the convective velocity, and C_p is the specific heat at constant pressure.

To estimate v , we consider the buoyancy force acting on the convective element. As the convective element rises it is lighter than its surroundings by

$$-(\delta \rho - \Delta \rho) = \frac{\left. \frac{\partial \rho}{\partial T} \right|_p}{\left. \frac{\partial \rho}{\partial p} \right|_T} (\delta T - \Delta T) = Q T (\nabla - \nabla_{ad}) \frac{\rho g}{p} \Delta z,$$

where Δz is the distance the convective element has travelled from its starting point. Hence the equation of motion for the convective element is

$$\rho \frac{d^2 \Delta z}{dt^2} = -(\delta \rho - \Delta \rho) g = Q T (\nabla - \nabla_{ad}) \frac{\rho g^2}{p} \Delta z.$$

Assuming that the mixing length is sufficiently small that everything except Δz in this equation is approximately constant, we can convert it into a conservation of energy equation by multiplying by $d\Delta z/dt$, and integrating with respect to time to get

$$v^2 = QT(\nabla - \nabla_{ad}) \frac{g^2}{\rho} l^2.$$

Putting the pieces together, we find that convective flux is given by

$$F_{conv} = Q^{1/2} C_p \left(\frac{T}{\rho} \right)^{3/2} (\nabla - \nabla_{ad})^{3/2} \rho^2 g^2 l^2.$$

This can be added to the radiative flux to give an equation for the structural gradient. The total luminosity carried by radiation and convection is

$$\begin{aligned} L &= 4\pi r^2 (F_{rad} + F_{conv}) = -\frac{16\pi a c r^2 T^3}{3\kappa\rho} \frac{dT}{dr} + 4\pi r^2 Q^{1/2} C_p \left(\frac{T}{\rho} \right)^{3/2} (\nabla - \nabla_{ad})^{3/2} \rho^2 g^2 l^2 \\ &= \frac{16\pi a c G m T^4}{3\kappa\rho} \nabla + 4\pi r^2 Q^{1/2} C_p \left(\frac{T}{\rho} \right)^{3/2} (\nabla - \nabla_{ad})^{3/2} \rho^2 g^2 l^2. \end{aligned}$$

The luminosity can be written in terms of the radiative gradient

$$L = \frac{16\pi a c G m T^4}{3\kappa\rho} \nabla_{rad},$$

to get

$$\nabla_{rad} = \nabla + A (\nabla - \nabla_{ad})^{3/2},$$

where

$$A = \frac{3\kappa p}{4acT^4} \rho^2 g l^2 Q^{1/2} C_p \left(\frac{T}{\rho} \right)^{3/2}.$$

It is convenient to introduce the quantity

$$\Gamma = \frac{\nabla_{rad} - \nabla}{\nabla - \nabla_{ad}},$$

which ranges from 0 to infinity depending on the efficiency of convection. In terms of this quantity

$$\nabla = \frac{\nabla_{rad} + \Gamma \nabla_{ad}}{1 + \Gamma},$$

and

$$\Gamma^3 + \Gamma^2 = A^2 (\nabla_{rad} - \nabla_{ad}),$$

which is easily solved for Γ .

This is the simplest form of mixing length theory. More sophisticated forms exist in which, for example, the convective element loses its excess energy as it moves rather than just at the end of its life. All that remains is to specify the mixing length. This is usually taken to be proportional to the pressure scale height

$$l = \alpha H_p = \alpha \left| \frac{dr}{d \ln p} \right| = \alpha \frac{p}{\rho g}.$$

The mixing length ratio, α , is fixed by calibrating with the properties of the Sun. It is often found to be about 1.5 (there is not a unique value for α because differences in treatment of the “physics” of the Sun can be compensated to some extent by changing α).

Note that 1.5 is not small compared to unity and hence the assumption that the properties of the surroundings are approximately constant over the mixing length is not strictly valid. This is one of the many deficiencies of mixing length theory.