PHYS 633 Introduction to Stellar Astrophysics Spring 2008

Homology relations

12.1 Introduction

Stars of similar mass and composition are expected to have similar physical condition in their interiors and hence are expected to have similar structure. Homology relations provide an approximate but useful way of investigating how the stellar structure depends on the stellar mass and composition.

12.2 Homology of ZAMS stars

Consider a star on the ZAMS. It will be in thermal equilibrium and will have essentially uniform composition. Depending on the star's mass, it will have convection zones in the center or in the surface layers. For the time being, we will ignore the complications that arise from convective energy transport and will assume that the star is radiative throughout. Using the mass as the independent variable, the equations of stellar structure are

$$\frac{dr}{dm}=\frac{1}{4\pi r^2\rho}, \tag{12.2.1}$$

$$\frac{dp}{dm}=-\frac{Gm}{4\pi r^4}, \tag{12.2.2}$$

$$\frac{dL}{dm}=\varepsilon, \tag{12.2.3}$$

and

$$\frac{dT}{dm}=-\frac{3\kappa L}{64\pi^2 acr^4T^3}. \tag{12.2.4}$$

In each equation, the right hand side is a product of a number of quantities. This suggests that we might be able to find *homology relations* that describe how the dependent variables scale with the total mass of the star, M. However, this will be possible only if the relations for ε, κ, ρ in terms of the dependent variables are also multiplicative in nature. Fortunately, in many situations, this is the case.

Often we are most interested in quantities at the stellar center or at the stellar surface. To derive the homology relations, we need as independent variable a quantity that gives the location of the center and surface in a way that is independent of M. Such a variable is the scaled mass,

$$q = \frac{m}{M}. \tag{12.2.5}$$

The center of the star is at $q = 0$ and the surface is at $q = 1$.

We now assume that the dependent variables are of form

$$\begin{aligned} r &= M^{a_r}\tilde{r}(q), \\ p &= M^{a_p}\tilde{p}(q), \\ L &= M^{a_L}\tilde{L}(q), \\ T &= M^{a_T}\tilde{T}(q), \end{aligned} \tag{12.2.6}$$

where the exponents a_r, a_p, a_L and a_T are all constants, and $\tilde{r}(q), \tilde{p}(q), \tilde{L}(q),$ and $\tilde{T}(q)$ are functions only of q.

From equation (12.2.1), we obtain

$$\frac{M^{a_r}}{M}\frac{d\tilde{r}}{dq} = \frac{1}{4\pi M^{2a_r}\tilde{r}^2\rho}, \tag{12.2.7}$$

so that

$$\rho = \frac{1}{M^{3a_r-1}}\frac{1}{4\pi\tilde{r}^2}\left(\frac{d\tilde{r}}{dq}\right)^{-1}. \tag{12.2.8}$$

Hence the scaling of the density with M is

$$\rho \propto \frac{1}{M^{3a_r-1}}. \tag{12.2.9}$$

From equation (12.2.2), we find

$$\frac{M^{a_p}}{M}\frac{d\tilde{p}}{dq} = -\frac{M}{M^{4a_r}}\frac{Gq}{4\pi\tilde{r}^4}, \tag{12.2.10}$$

so that

$$M^{4a_r+a_p-2} = -\frac{Gq}{4\pi\tilde{r}^4}\left(\frac{d\tilde{p}}{dq}\right)^{-1}. \tag{12.2.11}$$

Since the right hand side is independent of M, we find that

$$4a_r + a_p - 2 = 0. \tag{12.2.12}$$

Without going into the details, from equations (12.2.3) and (12.2.4), we find that the nuclear energy generation rate and opacity scale with *M* like

$$\varepsilon \propto M^{a_L-1}, \qquad (12.2.13)$$

and

$$\kappa \propto M^{4a_r+4a_T-a_L-1}. \qquad (12.2.14)$$

The relations in equations (12.2.9), (12.2.12), (12.2.13) and (12.2.14) are quite general because we have not yet made use of constitutive relations, i.e. the equation of state, opacity law or energy generation rate. We have one relation between the 4 homology exponents. We can find 3 more relations by specifying the constitutive relations. There are a number of simple yet physically relevant possibilities for these relations depending on the physical conditions in the stellar interior. For example for a low mass star, it is appropriate to use the perfect gas law as the equation of state, a Kramers' opacity law, and a power law approximation to the energy generation from the pp chains, whereas for a much more massive star it would be appropriate to use a radiation pressure equation of state, electron scattering opacity, and a power law approximation to the energy generation from the CNO cycles.

As an example consider the low mass star constitutive relations for which

$$\begin{aligned} p &\propto \rho T, \\ \kappa &\propto \rho T^{-7/2}, \\ \varepsilon &\propto \rho T^4. \end{aligned} \qquad (12.2.15)$$

Using (12.2.9) and the expression for *T* in (12.2.6), these give

$$\begin{aligned} p &\propto M^{a_T-3a_r+1}, \\ \kappa &\propto M^{-7a_T/2-3a_r+1}, \\ \varepsilon &\propto M^{4a_T-3a_r+1}. \end{aligned} \qquad (12.2.16)$$

Comparing with equations (12.2.13), (12.2.14) and the expression for *p* in equation (12.2.6), we get

$$\begin{aligned} & a_p = a_T - 3a_r + 1, \\ & 4a_r + 4a_T - a_L - 1 = -7a_T/2 - 3a_r + 1, \\ & a_L - 1 = 4a_T - 3a_r + 1. \end{aligned} \qquad (12.2.17)$$

Together with equation (12.2.12), we obtain a set of 4 simultaneous linear equations for the 4 homology exponents:

$$\begin{aligned} 4a_r &+ a_p & & &= 2, \\ 3a_r &+ a_p & - a_T & &= 1, \\ 14a_r & & +15a_T & -2a_L &= 4, \\ 3a_r & & -4a_T & + a_L &= 2. \end{aligned} \tag{12.2.18}$$

These have solution

$$a_r = 1/13,\, a_p = 22/13,\, a_L = 71/13,\, a_T = 12/13. \tag{12.2.19}$$

Note that for the effective temperature the scaling is given by

$$T_{eff} \propto \left(\frac{L_*}{R^2}\right)^{1/4} \propto M^{a_L/4 - a_r/2}, \tag{12.2.20}$$

and not a_T. For the above example, we find $T_{eff} \propto M^{5/4}$.

12.3 Sensitivity of stellar structure to nuclear reaction rate

We can use homology arguments to see how sensitive global properties of the star, such as its luminosity and radius, are to the energy generation rate. We can approximate the energy generation rate from hydrogen burning by

$$\varepsilon = \varepsilon_0 \rho T^{\eta}, \tag{12.3.1}$$

where ε_0 and η are constants. We can combine the electron scattering and Kramers' opacity laws into single expression by writing

$$\kappa = \kappa_0(\alpha)\left(\rho T^{-7/2}\right)^{\alpha}, \tag{12.3.2}$$

where $\alpha = 0$ for electron scattering and $\alpha = 1$ for Kramers opacity. The 4 equations for the homology exponents are now (assuming a perfect gas equation of state)

$$\begin{aligned} 4a_r &+ a_p & & &= 2 \\ 3a_r &+ a_p & - a_T & &= 1 \\ (8+6\alpha)a_r & & +(8+7\alpha)a_T & -2a_L &= 2(\alpha+1) \\ 3a_r & & -\eta a_T & + a_L &= 2 \end{aligned} \tag{12.3.3}$$

These have solution

$$\begin{aligned} a_r &= \frac{2\eta - 5\alpha - 2}{2\eta - \alpha + 6}, \\ a_p &= \frac{-4\eta + 18\alpha + 20}{2\eta - \alpha + 6}, \\ a_L &= \frac{(6+4\alpha)\eta + 13\alpha + 18}{2\eta - \alpha + 6}, \\ a_T &= \frac{4\alpha + 8}{2\eta - \alpha + 6}. \end{aligned} \tag{12.3.4}$$

We can write the exponent for the luminosity variable as

$$a_L = \frac{(6+4\alpha)\eta + 13\alpha + 18}{2\eta - \alpha + 6} = 3 + 2\alpha + \frac{2\alpha(\alpha+2)}{2\eta + 6 - \alpha}. \tag{12.3.5}$$

We see that for electron scattering opacity $a_L = 3$, independent of the value of η. For Kramers' opacity

$$a_L = 5 + \frac{6}{2\eta + 5}, \tag{12.3.6}$$

which only weakly depends on η. Hence the mass – luminosity relation is relatively insensitive to the temperature dependence of the nuclear energy generation rate. This is contrast to the pressure and temperature.

12.4 Sensitivity of stellar properties to composition

We can use a technique similar to homology analysis to estimate how the stellar properties depend on composition. To illustrate the method consider constitutive relations relevant to low mass stars. The pressure is given by the perfect gas law, which for small enough heavy element abundance and complete ionization, is

$$p = \frac{5X+3}{4}\frac{k}{m_u}\rho T. \tag{12.4.1}$$

The energy generation rate due to the pp chains is approximated by

$$\varepsilon = \varepsilon_0 X^2 \rho T^4. \tag{12.4.2}$$

The opacity is a little problematic because in the hot interior free – free opacity will be more important than bound – free opacity, whereas in the cooler outer layers the bound – free opacity will be larger. Although both opacities have a Kramers dependence on density and temperature, the dependence on composition is not the same. Since the heavy elements dominate the bound free opacity and also contribute significantly to the free – free opacity, we will take

$$\kappa = \kappa_0 Z(1+X)\rho T^{-7/2}. \tag{12.4.3}$$

Because of the different dependences on X and Z, we look for solutions of form

$$r = r_1(X)\, r_2(Z)\, r_3(m), \tag{12.4.4}$$

with similar expressions for the pressure, luminosity and temperature. (We do not need to introduce q, because we keep M fixed in this analysis.)

We find

$$\begin{aligned} r &\propto X^{4/13}(1+X)^{2/13}(5X+3)^{7/13} Z^{2/13}, \\ p &\propto X^{-16/13}(1+X)^{-8/13}(5X+3)^{-28/13} Z^{-8/13}, \\ L &\propto X^{-2/13}(1+X)^{-14/13}(5X+3)^{-101/13} Z^{-14/13}, \\ T &\propto X^{-4/13}(1+X)^{-2/13}(5X+3)^{-20/13} Z^{-2/13}. \end{aligned} \tag{12.4.5}$$

Note that these relations predict that stars of lower Z will be smaller, more luminous and hotter at the surface than stars of higher Z and the same mass (and hydrogen mass fraction). Since Z only enters through the opacity, these differences must be due to the lower opacity in stars of lower Z. A lower opacity means it is easier for radiation to leave the star and hence the star is more luminous. To generate the greater luminosity the central temperature must be higher. A central higher temperature leads to higher central pressure. To balance a larger pressure gradient, the gravity must be higher and hence the star must be smaller.

The same changes occur if Z is kept fixed and X is decreased. Hence fully mixed stars would evolve to the blue during hydrogen burning.

12.5 Stars with convective cores

Since the density is high, when convection occurs in the core of a star it is very efficient and so

$$\frac{d\ln T}{d\ln p} = \nabla_{ad}. \tag{12.5.1}$$

Applying the homology relations (12.2.6),

$$\frac{d\ln \tilde{T}}{d\ln \tilde{p}} = \nabla_{ad}, \tag{12.5.2}$$

independent of the values of a_p and a_T. Hence in the core there are too few equations to determine the homology exponents. However, continuity of temperature and pressure at the boundary between the convective core and the radiative envelope indicate that there is a solution in which a_p and a_T are determined by conditions in the radiative envelope and, assuming the constitutive relations do not change, the convective – radiative transition occurs at a value of q that is independent of M. Since massive stars have convective cores and radiative envelopes, these considerations show that homology arguments can be used for upper main sequence stars.

12.6 Stars with convective envelopes

Low mass stars, like the Sun, have convective envelopes. Because the convection results mainly from the high opacity associated with photo-ionization, the adiabatic gradient is not constant. Furthermore, in the outer parts of the convection zone, the density can be low enough that convection is not efficient. Hence the structural gradient can vary significantly with position in the convective envelope. This means that homology is not a good approximation for lower main sequence, as is evident from the large difference between the observationally determine mass – radius relation, $R \propto M$, and that obtained from homology arguments, $R \propto M^{1/13}$.