Superfluidity, Bose-Einstein condensation, and structure in one-dimensional Luttinger liquids

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(Received 26 July 2017; revised manuscript received 13 November 2017; published 19 January 2018)

We report diffusion Monte Carlo (DMC) and path integral Monte Carlo (PIMC) calculations of the properties of a one-dimensional (1D) Bose quantum fluid. The equation of state, the superfluid fraction $\rho_S/\rho$, the one-body density matrix $n(x)$, the pair distribution function $g(x)$, and the static structure factor $S(q)$ are evaluated. The aim is to test Luttinger liquid (LL) predictions for 1D fluids over a wide range of fluid density and LL parameter $K$. The 1D Bose fluid examined is a single chain of $^4$He atoms confined to a line in the center of a narrow nanopore. The atoms cannot exchange positions in the nanopore, the criterion for 1D. The fluid density is varied from the spinodal density where the 1D liquid is unstable to droplet formation to the density of bulk liquid $^4$He. In this range, $K$ varies from $K > 2$ at low density, where a robust superfluid is predicted, to $K < 0.5$, where fragile 1D superflow and solidlike peaks in $S(q)$ are predicted. For uniform pore walls, the $\rho_S/\rho_0$ scales as predicted by LL theory. The $n(x)$ and $g(x)$ show long range oscillations and decay with $x$ as predicted by LL theory. The amplitude of the oscillations is large at high density (small $K$) and small at low density (large $K$). The $K$ values obtained from different properties agree well verifying the internal structure of LL theory. In the presence of disorder, the $\rho_S/\rho_0$ does not scale as predicted by LL theory. A single $\nu_J$ parameter in the LL theory that recovers LL scaling was not found. The one body density matrix (OBDM) in disorder is well predicted by LL theory. The “dynamical” superfluid fraction, $\rho^D_S/\rho_0$, is determined. The physics of the deviation from LL theory in disorder and the “dynamical” $\rho^D_S/\rho_0$ are discussed.

DOI: 10.1103/PhysRevB.97.014513

I. INTRODUCTION

Dimensions play a key role in determining exotic phenomena such as Bose-Einstein condensation (BEC) and superfluidity. In 3D, BEC begins at a well-defined finite temperature [1–4] $T_{\text{BEC}}$. Below $T_{\text{BEC}}$, a finite fraction of the bosons occupy one single particle state and the one-body density matrix (OBDM) develops a long-range tail [4–6]. In bulk, uniform systems, the tail is flat and extends to infinity. Superfluidity (and superconductivity) can be formulated [7] as arising from the coherent field established by the macroscopically occupied state. $T_s$ for superflow equals $T_{\text{BEC}}$. Below $T_s$, the superfluid fraction, $\rho_S/\rho$, increases smoothly with decreasing temperature, from zero at $T_s$ to unity at 0 K.

In 2D systems at low temperature, the OBDM develops a tail that decays algebraically with distance. In 2D liquid $^4$He the algebraic tail can extend [6] to over 100 Å but is not macroscopic. A $T_{\text{BEC}}$ in 2D can be defined as the temperature at which the OBDM first develops this algebraic tail. Superfluidity in 2D is generally formulated [8,9] independently of BEC. There is an abrupt jump in $\rho_S/\rho$ from zero to a finite fraction [9] at $T_s$, in contrast to 3D. $T_s$ can also be identified [8] from the OBDM as the temperature at which the exponent that describes the algebraic tail reaches a specific value.

In 1D spinless Bose and Fermi fluids, low energy phenomena such as superflow and BEC is expected to follow the predictions of Luttinger liquid (LL) theory [10]. The OBDM is predicted to show a decaying algebraic tail at low temperature that has oscillations which extend to large distances reflecting the atomic order in 1D. The pair distribution function (PDF) has similar long-range oscillations. These oscillations are uniquely characteristic of 1D and are not seen in the OBDM or PDF in 2D or 3D fluids. Superfluidity in 1D is really only a finite size effect. There is no $T_c$. The $\rho_S/\rho_0$ has a characteristic 1D shape and scales as $LT$, the product of the temperature $T$ and length $L$ of the 1D system.

There has been an extensive search for 1D quantum fluids. $^4$He or other bosons confined in small diameter nanopores may show 1D behavior [11–17]. Most of the $^4$He in the nanopores is deposited in layers (solid and liquid) on and near the pore walls [11–16,18,19]. These layers have properties characteristic of 2D or 3D. Those atoms confined to the center or core of the nanopore could be a 1D system [20–22]. $^4$He confined to a dislocation in solid helium is also predominantly a 1D system [23]. $^4$He and other atoms confined in and on carbon nanopore bundles can show 1D character [24].

Cold atoms in highly asymmetric optical lattices can be tightly restricted in two dimensions but highly extended in the third dimension [25,26]. While atomic displacements in all directions are possible, exchange and tunneling between lattice sites are predominantly along the extended 1D. The central goal of this exciting field is to explore transitions from superfluidity to Mott-insulator [27,28] or Bose glass (BG) [29–32] states, and to determine critical momentum [33] and phase slips [34] in 1D. Superconductivity in 1D is also anticipated in nanowires [35]. Specifically, the role of phase slips in loss of superflow in nanowires has been extensively investigated [36]. The theory of 1D systems has been equally extensively developed [37–42].

The goal of the present paper is to display the properties of a genuine 1D Bose quantum fluid. It is particularly to test whether the properties are well predicted by LL theory. We
consider liquid $^4$He confined to a line in a narrow nanopore of radius $R = 3$ Å. The nanopore wall represents the amorphous solid layers of $^4$He (~1.5 layers) on the interior of the nanopore as well as the nanopore [43]. We present DMC calculations of the equation of state (EOS), the pair distribution function (PDF) $g(x)$, and the static structure factor $S(q)$, and PIMC calculations of the superfluid fraction $\rho_s/\rho_0$, and the one-body density matrix (OBDM) $n(x)$, of the confined liquid. The aim is to verify LL predictions as a function of the 1D density and the value of the LL parameter $K$ which is a sensitive function of density. It is also to display clearly the properties of 1D fluids as a function of density. It is also to verify LL predictions as a function of the 1D density and $g(x)$ which is a sensitive function of density. We also evaluate the "dynamical" superfluid fraction defined in Refs. [44,45].

Liquid $^4$He at reduced dimension has been investigated recently using PIMC methods. Del Maestro and Affleck [46] explored pure 1D helium, a line of density was set at the bulk liquid density. For liquid pore radii the liquid in the nanopore need be simulated using PIMC. The possible superfluid. As for liquid $^4$He, the search includes $p$-$^2$H and $p$-$^3$H as discussed in Sec. IV. For low energy, long wavelength fluctuations in the density $\rho(x)$, Haldane [10] shows that $H$ may be simplified to

$$H_{LL} = \frac{\hbar}{2\pi} \int dx [v_J(\nabla \phi)^2 + v_N(N - \rho_0)^2]$$

$$= \hbar \left\{ \sum_{q \neq 0} \omega_q b_q^\dagger b_q + \left( \frac{\pi}{2L} \right) [v_J J^2 + v_N(N - N_0)^2] \right\}.$$  

(1)

$H_{LL}$ describes an effective harmonic fluid. The term $\hbar v_J(\nabla \phi)^2$ represents the kinetic energy density. The density is $\rho(x) = \rho_0 + \Pi(x)$ where $\Pi(x)$ is a small and long wavelength change in $\rho(x)$ and $\nabla \phi = \pi [\rho_0 + \Pi(x)]$. The term $\hbar v_N(N - \rho_0)^2$ represents the potential energy density of the density fluctuations. The $\omega_q$ are the frequencies of the density modes. Since $q = 2\pi/k$ is small, $\omega_q = v_3 q$ where $v_3$ is the sound velocity. The index $J$ denotes the number of phase twists in $\phi$ over the length $L$, $\phi(x) = 2\pi J$, $N - N_0$ is the deviation of $N$ from the average $N_0 = \rho_0 L$. We consider a cylindrical ensemble in which $N$ is fixed at $N_0$.

For a Galilean invariant, 1D LL $h v_0^j = \pi(h^2/m)\rho_0$ and $h v_3 = (\pi\rho_0^2/k)^{-1}$ where $k$ is the 1D compressibility. At $T = 0$ K and expressing $V = AL$, the 1D compressibility is $\kappa^{-1} = AK^{-1}$ where $\kappa^{-1} = -\nu(\partial\rho/\partial V)_T$ is the 3D compressibility. In terms of the 1D pressure $P = A\rho$,

$$\kappa^{-1} = \rho_0 \frac{\partial P}{\partial \rho_0},$$

(2)

where $P = \rho_0^2 \partial e/\partial \rho_0$ and $e/E/N$. Since $K = (v_3^2/v_N)^{1/2} = \pi^{-3/2}(m\rho_0^2)^{1/2}$, large $K$ corresponds to large $K$, e.g., a gaslike fluid. For $K > 2$ and $K > 3/2$ a superfluid robust to a periodic external potential and to disorder, respectively, is predicted [52]. Small $K$ corresponds to small $K$, e.g., a solidlike fluid at high density where superflow is fragile.

The sound velocity of the 1D LL density modes, given by the usual expression $v_3^2 = (m\rho_0^2k)^{-1}$, can be written as $v_3 = (v_0^2/v_N)^{1/2}$. The parameter $K$ is often expressed as $K = v_3^2/v_N$ and $K = v_3/v_N$. The physics of 1D systems has been extensively reformulated and reviewed [37–39]. From $H_{LL}$, Haldane and others make predictions that we denote LL predictions. For the pair distribution function (PDF),

$$g(x) = \frac{1}{\rho_0^2} \langle \rho(x) \rho(0) \rangle$$

$$= 1 - 2K(2\pi\rho_0 x)^{-2}$$

$$+ \sum_{n=1}^{\infty} (\rho_0 x)^{-2} e^{2\pi n x} A_n \cos(2\pi n x).$$

(3)

**II. LUTTINGER LIQUID PREDICTIONS**

In this section we outline the predictions made from LL theory. Haldane [10] and others [37–39,46] consider a 1D line of $N$ bosons of average density $\rho_0 = N_0/L$ with periodic boundary conditions over a length $L$. The field operator $\Psi(x)$ in the general Hamiltonian $H$ is expressed as a magnitude and a phase, $\phi(x)$, $\Psi(x) = \rho(x)^{1/2} e^{i\phi(x)}$. For low energy, long wavelength fluctuations in the density $\rho(x)$, Haldane [10] shows that $H$ may be simplified to

$$H_{LL} = \frac{\hbar}{2\pi} \int dx [v_J(\nabla \phi)^2 + v_N(N - \rho_0)^2]$$

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$$+ \sum_{n=1}^{\infty} (\rho_0 x)^{-2} e^{2\pi n x} A_n \cos(2\pi n x).$$

(3)
where $L$ follows from the general relation between superfluid fraction $ho_S/\rho$ and the equilibrium state $e(\rho_0) = E(\rho_0)/N$ using Eq. (2), by fits of the LL expression Eq. (3) to the calculated $g(x)$ and from $q \to 0$ limit of $S(q)$, Eq. (10), and (c) from fits of the LL expression Eq. (4) to $n(x)$. If the predictions of LL theory are internally consistent, the $K$ values obtained from these three routes should agree.

The expression of state $E(\rho_0)$ at $T = 0$ K was calculated using DMC methods as were $g(x)$ and $S(q)$ using the expressions,

$$g(x) = \frac{1}{\rho_0 N} \sum_{i,j} \delta(x - |x_i - x_j|),$$

and

$$S(q) = \frac{1}{N} \sum_{i,j} e^{-i q (x_i - x_j)}.$$

The OBDM, $n(x)$, and the winding number $\langle W^2 \rangle$ at finite temperature were calculated using the worm algorithm PIMC and standard methods [6]. To restate, we used the canonical ensemble version of the worm algorithm PIMC code in which $N$ is fixed at $N = N_0$; $\rho_0 = N/L = N_0/L$.

To close this section, we note that a “dynamical” superfluid fraction, $\rho_S^D/\rho_0$, has been introduced [45,54]. This is in addition to the “thermodynamic” $\rho_S/\rho_0 = \tilde{\Phi}(F_0/N)/\tilde{\Phi}(1/2 m v_F^2) = \alpha_0(\rho_0^D)$, defined previously that we were using PIMC. The “thermodynamic” $\rho_S^D/\rho_0$ is given [45,53] by the change in Helmholtz free energy arising from a superfluid velocity $v$. The defining relation for $\rho_S^D/\rho_0$ may be taken as [45]

$$F_{\text{eff}} = \frac{\hbar^2}{2 m \rho_S^D} \int dx (\nabla \Phi)^2,$$

where $\Phi$ is the phase of the order in the liquid which explicitly includes phase twists over the length $L$. Prokof’ev and Svistunov [45] show that, from $F_{\text{eff}}$, the probability of observing a number of phase twists $J$ and winding number $W$ are both Gaussian in $J$ and $W$ as obtained in Ref. [46] from $H_{\text{LL}}$. Specifically, the probability of observing a $W$ obtained from $F_{\text{eff}}$ is [45]

$$P^D(W) \propto \exp \left( -\frac{1}{2} \frac{L T}{\sigma \rho_S^D} W^2 \right) = \exp \left( -\frac{1}{2} \alpha^D W^2 \right)$$

in 1D. Equation (14) is the same as the LL $P(W)$ in Eq. (9) with $\alpha$ replaced by $\alpha^D$ where

$$\alpha^D = \alpha_0 \left( \frac{\rho_S^D}{\rho_0} \right)^{-1}.$$

From the general relation $\rho_S/\rho_0 = \alpha_0(\rho_0^D)$ and using $P^D(W)$ in Eq. (14) to evaluate $\langle W^2 \rangle$ we obtain Eq. (6) with $\alpha$ replaced by $\alpha_D$, i.e.,

$$\rho_S/\rho_0 = \alpha_0 \left( \frac{\rho_S}{\rho_0} \right)^{-1} \left[ \Theta_3(0, e^{-\alpha_0^2/2}) \right] \theta_3(0, e^{-\alpha_0^2/2}).$$

This provides a relation [45,54] between $\rho_S/\rho_0$ and $\rho_S^D/\rho_0$.

In a uniform system (Galilean invariant), if LL predictions are correct, we expect $P(W)$ in Eq. (9) to be correct with $v_j = v_j^0$ and $\alpha = \alpha_0$. Comparing Eqs. (9) and (8) with Eqs. (15)
and (14) requires \( \rho_S^D / \rho_0 = 1 \) for a uniform system. In disorder or in a periodic external potential, \( v_j \) can differ from \( v_j^0 \). However, if the LL predictions hold with \( v_j \) a constant, then we expect \( \rho_S^D / \rho_0 \) to be a constant independent of \( T \) or \( L \), i.e., \( \rho_S^D / \rho_0 = v_j / v_j^0 \) to be constant. In disorder we can fit Eq. (16) to the PIMC \( \rho_S / \rho_0 \) and obtain \( \rho_S^D / \rho_0 \). Indeed Machta and Guyer [54] and Prokof'ev and Svistunov [45] have proposed this fit to obtain \( \rho_S^D / \rho_0 \). To do this, they propose a limit of Eq. (16) valid at large \( L/T \). The limit is obtained by keeping only \( W = \pm 1 \) in the numerator and \( W = 0 \) in the denominator in Eq. (16). So that Eq. (16) reduces to

\[
\frac{\rho_S}{\rho_0} = \alpha^0_2 \exp \left[ -\frac{1}{2} \alpha^0_1 \left( \frac{\rho_S^D}{\rho_0} \right)^{-1} \right]. \tag{17}
\]

The full expression Eq. (16) or the limit Eq. (17) proposed in Refs. [45,54] can be fitted to the PIMC \( \rho_S / \rho_0 \) to determine \( \rho_S^D / \rho_0 \), as we do in the present paper.

### III. MODEL OF THE NANOPORE AND METHODS

We simulate liquid \(^4\)He confined in a nanopore described by the Hamiltonian,

\[
\hat{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(r_{ij}) + \sum_{i=1}^{N} V(\rho_i), \tag{18}
\]

where \( N \) is the number of \(^4\)He atoms of mass \( m \), \( U(r) \) represents the interaction between \(^4\)He atoms modeled by the Aziz potential [55], and \( V(\rho) \) is the confining potential at a distance \( \rho \) from the center of the pore. To reduce the system size, we include the effect of the amorphous solid \(^4\)He layers on the interior walls of the nanopore in the external potential \( V(\rho) \). Our model is thus exactly the same as that used previously in Ref. [43].

The walls of real porous media plus solid \(^4\)He layers are rough and irregular. This leads to a disordered confining potential for the liquid helium. In order to model the irregularity of the potential we have, as in Ref. [43], added fictitious fixed particles to the nanopores at random positions along the whole length of the pore and at all angles, separated from 2.5 Å to 3.5 Å from its axis. The average linear density of the fixed particles was 1 Å \(^{-1} \). We represented their interaction with the helium atoms by a Lennard-Jones potential with parameters \( \epsilon_d = 1.5 \) K and \( \sigma_d = 2 \) Å. The effect of this external potential is the creation of random pockets of stronger attraction or repulsion. For a given length \( L \) of the nanopore, the impurity potential is the same for all densities of the \(^4\)He in the pore.

Finite temperature calculations are performed using the finite temperature worm algorithm path-integral Monte Carlo [6,56]. The values of the discretized imaginary time \( \delta \tau \) were the same as in Ref. [43], i.e., 0.004 K \(^{-1} \). For calculations at zero temperature the second-order diffusion Monte Carlo method was used [57], which solves stochastically the Schrödinger equation written in imaginary time. The details of the method are given in Ref. [57]. The guiding wave function was as usual constructed as \( \Psi = \prod_{i<j} f(r_{ij}) \phi(\rho_i) \), where \( f(r) = \exp(-b/r^2) \) and \( \phi(\rho) \) was the exact single-particle solution in the confining potential \( V(\rho) \). Care was taken to eliminate any residual biases, such as the time-step or the population-size bias.

### IV. RESULTS

#### A. Uniform nanopores

1. **Bulk liquid \(^4\)He density**

   Figure 1 shows the superfluid fraction, \( \rho_S / \rho_0 \), the OBDM, \( n(x) \), and the pair distribution function, \( g(x) \), of 1D liquid \(^4\)He confined in a uniform nanopore of radius \( R = 3 \) Å at bulk liquid \(^4\)He density. The corresponding linear density of the \(^4\)He is \( \rho_0 = 0.267 \text{ Å}^{-1} \), interatomic spacing \( a = \rho_0^{-1} = 3.75 \) Å, along the nanopore. At \( R = 3 \) Å, it is energetically very difficult for two \(^4\)He atoms to exchange positions in the pore. In this sense it is a 1D system.

   In Figs. 1(a) and 1(b), \( \rho_S / \rho_0 \) and \( n(x) \) calculated using PIMC are reproduced from Ref. [43]. The solid line in Fig. 1(a) is a fit of the LL prediction for \( \rho_S / \rho_0 \), Eq. (6), with the LL parameter \( \hbar \nu_x \) treated as a free fitting parameter. The best fit value of \( \hbar \nu_x \) agrees with LL expression of a uniform system, \( \hbar \nu_j^0 = \pi(h^2/m)\rho_0 = 10.16 \text{ Å} K \), within fit precision. The fit is good with a \( \chi^2 \) close to unity. The PIMC \( \rho_S / \rho_0 \) clearly scales as the product \( LT \) as predicted for 1D by LL theory. In Fig. 1(b), the solid line is a fit of the LL prediction, Eq. (4), to the PIMC \( n(x) \). Only the first term in the series for \( n(x) \) was retained with \( B_1 \) and \( K \) as free fitting parameters. The fit of the LL prediction to the PIMC \( n(x) \) is clearly good.

   In Fig. 1(c), the dashed line show the fit of the LL prediction, Eq. (3), to \( g(x) \) calculated using DMC. Again only the first term in the series was retained with \( A_1 \) and \( K \) treated as free fitting parameters. The fit is excellent and the \( K \) value obtained from the fit agrees well with \( K \) calculated from the equation of state and \( S(q) \). The \( g(x) \) clearly shows the long range oscillations characteristic of 1D at a linear density of \( \rho_0 = 0.267 \text{ Å}^{-1} \).

   Equations (3) and (4) are valid for a LL liquid with periodic boundary conditions (BCs) over a length \( L \rightarrow \infty \). For finite \( L \), the \( \rho_0 x \) in the denominators of the terms in Eqs. (3) and (4) should be replaced by \( \rho_0 x \rightarrow \rho_0 (L/\pi) \sin(\pi x / L) \) (see Eqs. 61 and 62) of Cazalilla [58]. We actually show the finite \( L \) expressions in Fig. 1 but the difference from Eqs. (3) and (4) is, except very near \( L/2 \), hardly perceptible on the scales shown in Figs. 1(b) and 1(c).

2. **Density dependence**

   We now explore the density dependence of 1D liquid \(^4\)He. Figure 2 shows the energy per particle, \( E(\rho_0) = E(\rho_0)/N \), the pressure, \( p(\rho_0) \), and the sound velocity, \( v_S(\rho_0) \), of liquid \(^4\)He in the uniform nanopore (\( R = 3 \) Å) versus the linear density, \( \rho_0 \). As the temperature is lowered the finite temperature PIMC results approach the \( T = 0 \) K DMC values. \( E(\rho_0)/N \) has a broad minimum at \( \rho_0 = 0.108 \text{ Å}^{-1} \). Specifically, the pressure is zero at \( \rho_0 = 0.108 \text{ Å}^{-1} \) showing that the equilibrium density in the nanopore is significantly lower than the linear density corresponding to bulk liquid \(^4\)He, \( \rho_0 = 0.267 \text{ Å}^{-1} \). There is a modest pressure in the nanopores, \( p \gtrsim 35 \) bar, at \( \rho_0 = 0.267 \text{ Å}^{-1} \). The sound velocity goes to zero at \( \rho_0 = 0.09 \text{ Å}^{-1} \), which defines the spinodal density where the 1D liquid is...
with \( \bar{T} \) the best fit value of \( R \) confined in a uniform nanopore of radius \( \rho \). The 3D liquid density in the pore, \( \rho_0 = 0.267 \AA^{-1} \), is set close to the density of bulk liquid at saturated vapor pressure (SVP), \( \rho = 0.0218 \AA^{-1} \). The PIMC data for \( \rho_0/\rho_0 \) and \( n(x) \) are from Ref. [43]. The LL prediction for \( \rho_0/\rho_0 \) is fitted to the PIMC \( \rho_0/\rho \) with \( \hbar v \) as a free fitting parameter. The error of the individual \( \rho_0/\rho_0 \) values is included in the fit. The fit is good with a \( \chi^2 \) near unity. The best fit value of \( \hbar v \) agrees with the uniform liquid, LL value, \( \hbar v_0 = \pi (\hbar^2/m) \rho_0 = 10.17 \) K \AA, within the fit error. The PIMC \( n(x) \) and the DMC \( g(x) \) show long range oscillations as predicted by LL theory, oscillations that are unique to 1D fluids.

mechanically unstable. At the spinodal density, the uniform liquid appears to separate into linear droplets. The spinodal density lies very close to the equilibrium density.

FIG. 1. The (a) superfluid fraction, \( \rho_S/\rho_0 \), (b) one body density matrix, \( n(x) \), and (c) pair distribution function, \( g(x) \), of liquid \(^4\)He confined in a uniform nanopore of radius \( R = 3 \) Å. The 3D liquid density in the pore, \( \rho_0 = 0.267 \AA^{-1} \), is set close to the density of bulk liquid \(^4\)He at saturated vapor pressure (SVP), \( \rho = 0.0218 \AA^{-1} \). The PIMC data for \( \rho_0/\rho_0 \) and \( n(x) \) are from Ref. [43]. The LL prediction for \( \rho_0/\rho_0 \) is fitted to the PIMC \( \rho_0/\rho \) with \( \hbar v \) as a free fitting parameter. The error of the individual \( \rho_0/\rho_0 \) values is included in the fit. The fit is good with a \( \chi^2 \) near unity. The best fit value of \( \hbar v \) agrees with the uniform liquid, LL value, \( \hbar v_0 = \pi (\hbar^2/m) \rho_0 = 10.17 \) K \AA, within the fit error. The PIMC \( n(x) \) and the DMC \( g(x) \) show long range oscillations as predicted by LL theory, oscillations that are unique to 1D fluids.

\[ \bar{T} \]

\[ \text{FIG. 2. (a) The energy per boson} \ e(\rho_0) \text{ versus linear density} \ \rho_0 \text{ of 1D liquid} \ ^4\text{He in a nanopore of radius} \ R = 3 \ \text{Å.} \ e(\rho_0) = E(\rho_0)/N \text{ has a broad minimum at} \ \rho_0 \approx 0.108 \ \text{Å}^{-1}. \text{ (b) The pressure versus} \ \rho_0. \text{ The pressure is zero at} \ \rho_0 = 0.108 \ \text{Å}^{-1}. \text{ At} \ \rho_0 = 0.267 \ \text{Å, the 3D pressure is} \ p \approx 35 \ \text{bar}. \text{ (c) The sound velocity,} \ v_S \text{ versus} \ \rho_0. \text{ The LL parameter is} \ K = v_S^2/v_g. \text{ The sound velocity vanishes at} \ \rho_0 = 0.09 \ \text{Å}^{-1} \text{ which defines the spinodal density where the 1D liquid becomes mechanically unstable to droplet formation.} \]

\[ \text{FIG. 3.} \text{ The Luttinger liquid parameter} \ K \text{ of the 1D liquid versus linear density} \ \rho_0. \text{ The} \ K \text{ calculated from the equation of state (the sound velocity), the static structure factor,} \ S(q), \text{ and from fits to the} \ g(x) \text{ agree well, e.g., within a few percent at} \ \rho_0 = 0.25 \ \text{Å}^{-1}. \text{ The} \ K \text{ obtained from} \ g(x) \text{ is less well determined at lower densities since the amplitude of the oscillations in} \ g(x) \text{ is smaller at lower densities and the linear region of} \ S(q) \text{ is reached only at low} \ q. \text{} \ K \text{ varies from} \ K \geq 2 \text{ at and below the equilibrium density} \ \rho_0 = 0.108 \ \text{Å}^{-1} \text{ to} \]

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The Luttinger liquid parameter $K = (v_J^0/v_N)^{1/2}$ versus density $\rho = N/L$ at between $\rho_0 = 0.10 \, \text{Å}^{-1}$ and $\rho_0 = 0.30 \, \text{Å}^{-1}$ ($\rho_0 = 0.090 \, \text{Å}^{-1}$ is the spinodal density and $\rho_0 = 0.267 \, \text{Å}^{-1}$ is the linear density corresponding to bulk liquid $^4$He density at SVP). The solid line is $K$ calculated from the equation of state. The solid circles and open squares are $K$ calculated from the static structure factor $S(q)$ and pair correlation function $g(x)$, respectively. $K$ varies from $K \gtrsim 2.0$ at the equilibrium density, $\rho_0 = 0.108 \, \text{Å}^{-1}$, to $K < 0.5$ at $\rho_0 = 0.267 \, \text{Å}^{-1}$.

$K < 0.5$ at $\rho_0 > 0.23 \, \text{Å}^{-1}$. Essentially at lower density, where the interaction is weaker, the kinetic energy ($v_J$) becomes relatively more important than the potential energy ($v_N$). $K = (v_J^0/v_N)^{1/2}$ is a sensitive function of linear density $\rho_0 = N/L$.

Figure 4 shows the PIMC superfluid fraction, $\rho_S/\rho_0$, at four linear densities between $\rho_0 = 0.25 \, \text{Å}^{-1}$ and $\rho_0 = 0.10 \, \text{Å}^{-1}$, the latter close to the spinodal density $\rho_0 = 0.09 \, \text{Å}^{-1}$. At the higher densities, $\rho_0 = 0.20 \, \text{Å}^{-1}$ and $\rho_0 = 0.25 \, \text{Å}^{-1}$, the LL prediction fits the PIMC $\rho_S/\rho_0$ well, the $\chi^2$ is close to unity, and the best fit value of $h v_J$ agrees with the uniform system value $h v_J^0$, as in Fig. 1(a). However, at lower density, $\rho_0 = 0.133 \, \text{Å}^{-1}$ and $\rho_0 = 0.10 \, \text{Å}^{-1}$, the $\chi^2$ is significantly larger. At lower density there does not appear to be a systematic deviation of the PIMC $\rho_S/\rho_0$ from the LL predictions. Rather there is a wider scatter of the PIMC $\rho_S/\rho_0$. The wider scatter may arise because at low density we have fewer atoms (lower statistics) in a given length $L$ and $L$ cannot be too long or there will be little or no winding. In addition, the density $\rho_0 = 0.10 \, \text{Å}^{-1}$ is close to the spinodal density and in the PIMC simulations we observe periods of separation of the liquid into droplets during the simulation, in which case LL scaling is not expected.

Figure 5 shows the OBDM, $n(x)$, calculated using PIMC methods at five linear densities between $\rho_0 = N/L = 0.117 \, \text{Å}^{-1}$ and $\rho_0 = 0.25 \, \text{Å}^{-1}$. As in Fig. 1(b), the $n(x)$ initially drops rapidly over a short distance $x$. This reflects the high localization of the atoms in real space (a wide momentum distribution) especially at higher density. The large amplitude oscillations in $n(x)$ at large $x$ at $\rho_0 = 0.25 \, \text{Å}^{-1}$ arise from the atomic correlations at high density. The fits to the PIMC OBDM shown (dotted lines) are Eq. (4) modified for the finite $L$ periodic boundary conditions used in PIMC (see Eq. (62) of Ref. [58]). The LL parameter $K$ obtained from the fit is consistent with $K$ obtained from the equation of state and $S(q)$. The dashed-dotted line is Eq. (4) which shows that using the finite $L$ expressions [58] makes little difference to the OBDM in this case.

Cazalilla [58] has derived expressions for correlation functions, $g(x)$ and $n(x)$, valid at finite temperature in the limit of an infinite box size. These expressions show that the $T = 0$ K limit where the correlations functions show algebraic decay is reached only at lengths less than a thermal correlation length $L_T = h v_J/(k_B T)$, i.e., at $x \ll L_T$. At finite temperature and long lengths $x \gtrsim L_T$, $g(x)$ and $n(x)$ cross over to exponential decay. At the three highest densities shown in Fig. 5, $L_T \gtrsim 30 \, \text{Å}$, so that finite temperature exponential scaling cannot be observed for the $x \lesssim 30 \, \text{Å}$ shown. At $\rho_0 = 0.117 \, \text{Å}^{-1}$, $L_T \approx 9.4 \, \text{Å}$, and the heavy dotted line shows a fit of the finite temperature expression (Eq. (71) of Ref. [58]) to the PIMC OBDM. The finite $T$ expression clearly fits the data well. We could not use the whole range up to half the box size, because the periodic boundary conditions cause the OBDM to have zero slope at half the box size, and in this particular case the slope started to change almost $10 \, \text{Å}$ before $L/2$. For the density $0.133 \, \text{Å}^{-1}$ no fit is provided because $L_T \approx 14 \, \text{Å}$ so the range for both algebraic and exponential decay is too short.

The DMC pair distribution function versus linear density is shown in Fig. 6(a) for densities $0.1 \leq \rho_0 \leq 0.3 \, \text{Å}^{-1}$. While the amplitude of the oscillations decreases with decreasing density, there remain clearly observable oscillations at low density. The LL prediction for $g(x)$ fits the DMC $g(x)$ well down to the lowest densities.

Algebraic decay of $g(x)$ at $T = 0$ K is expected to cross over to exponential decay at finite temperature, as observed in $n(x)$. In $g(x)$ the cross over is most clearly observed in the decay of the oscillations. Figure 6(b) shows a zoomed pair distribution function at two temperatures. The more rapid decay at higher temperature is most clearly seen at the highest density shown ($0.250 \, \text{Å}^{-1}$), where the thermal length $L_T \approx 54 \, \text{Å}$ at 0.5 K and $L_T \approx 18 \, \text{Å}$ at 1.5 K. The same type of behavior is present at all densities, but less visible.

Figure 7 shows the static structure factor, $S(q) = \frac{1}{L^2} \delta(\rho_0) \rho_0^{-1} \langle \langle \hat{\rho} \delta(x-x) \rangle \rangle$ at $T = 0$ K. $S(q)$ is shown for liquid $^4$He in the nanopore at linear densities, $0.117 < \rho_0 < 0.30 \, \text{Å}^{-1}$. $S(q)$ is dominated by a single peak at $q = 2 \pi \rho_0$. The peak height and sharpness of the peak increases with increasing density $\rho_0$. A small second peak emerges at $q = 4 \pi \rho_0$ at the highest densities $\rho_0$. The dominant peak at $q = 2 \pi \rho_0$ arises from the nearly solidlike order in the 1D liquid and is characteristic of 1D. Using the relation,

$$\frac{1}{2\pi} \int dq e^{-i q x} [S(q) - 1] = \rho_0 [g(x) - 1], \quad (19)$$

we see that the dominant single peak in $S(q)$ at $q = 2 \pi \rho_0$ will lead to oscillations in $g(x)$ dominated by a single term $A_1 \cos(2 \pi \rho_0 x)$. Thus the long range oscillations in $g(x)$ and sharp peak in $S(q)$ are complimentary characteristics of a highly correlated 1D fluid. Within precision we found that $S(q)$ was linear in $q$ at $q \to 0$, although a low $q$ value is needed to reach the linear region at low $\rho_0$. Using the relation (10), we may see visually from Fig. 7 that the LL $K$ decreases with increasing $\rho_0$. 

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uniform nanopore (radius $\rho$) for superfluidity, Bose-Einstein condensation, etc. [1].

There does not appear to be accurate $\rho_S/\rho$ especially at higher linear density, $3 \AA$. These properties were well described by LL expressions $\rho_S/\rho$.

In the nanopore containing point disorder in the walls as well as $\rho_S/\rho$ at all densities. The fits of the LL expression $\rho_S/\rho$ of the 1D liquid $\rho_S/\rho$ to the LL prediction $\rho_S/\rho$ at $0$ for liquid $4\text{He}$ at five linear densities $\rho_S/\rho$.

$\chi$ is large even at high density $\rho_S/\rho$ for the three different lengths $L$ as is the uniform nanopore. At high density, the longer $L$ appear to be associated with lower $\rho_S/\rho$, lower than predicted by Eq. (6).

If we interpret the PIMC $\rho_S/\rho$ in terms of the LL Hamiltonian, then the fit shown in Fig. 8 is a fit of Eq. (6) with $v_J/v_J^0$ in Eq. (8) as the fitting parameter. The best fit ratio $v_J/v_J^0$ is shown at each density in Fig. 8. Also, since from Eq. (6) $\rho_S/\rho = v_J/v_J^0$ at $T = 0 \text{ K}$, the value of $v_J/v_J^0$ can be read directly from $\rho_S/\rho$ at $T = 0 \text{ K}$ in Fig. 8.

In disorder, LL theory predicts that $\rho_S/\rho$ should continue to scale with $LT$ but with a constant $\hbar v_J$ that is different from $\hbar v_J^0$ and characteristic of the disorder. We note firstly that $v_J/v_J^0$ is not accurately determined. For example, the low values of $v_J/v_J^0$ at some densities in Fig. 8 appear to be associated with a lack of data at low $LT$ where $\rho_S/\rho$ may go toward unity. Also as the homogeneous case, at the density $\rho = 0.1 \AA^{-1}$, occasional separation into droplets in the course of the simulation is observed, which contributes to the low quality of the fit. Considering the remaining four densities, a value of $v_J/v_J^0$ at $85 \pm 2 \AA$ is obtained, however with the $\chi^2$ close to 0. Excluding the data for the shortest length $L = 15 \AA$, a $v_J/v_J^0 = 0.86 \pm 0.02$ would be obtained with $\chi^2 \approx 50$ at $\rho_J = 0.2 \AA^{-1}$.

The impact of disorder on $\rho_S/\rho$ is predicted [52] to be a sensitive function of $K$. In the present 1D $^4\text{He}$, $K$ depends...
on $\rho_0$ as shown in Fig. 3. For $K \gtrsim 3/2$ 1D superfluidity is predicted to be robust in disorder. For $K \lesssim 3/2$ ($\rho_0 \lesssim 0.14$ Å$^{-1}$) small amplitude disorder is predicted [52] to destroy 1D superfluidity. The suppression of $\rho_S/\rho_0$ at $K \lesssim 3/2$ is observed here only in the longer length samples investigated. At $\rho_0 = 0.2$ Å$^{-1}$ and 0.25 Å$^{-1}$ where $K < 3/2$ a small $\rho_S/\rho_0$ (near zero at $\rho_0 = 0.2$ Å$^{-1}$) is indicated at the longest $L$ only. This suggests that $\rho_S/\rho_0$ can be suppressed to zero by disorder at $K \lesssim 3/2$ but a long sample is needed to reveal the suppression.

If we interpret the PIMC $\rho_S/\rho_0$ in Fig. 8 as an opportunity to determine the “dynamical” superfluid fraction $\rho_{S}^{D}/\rho_0$, then the fit in Fig. 8 is a fit of Eq. (16) with the “dynamical” $\rho_{S}^{D}/\rho_0$ in Eq. (15) as a free fitting parameter. The best fit values of $\rho_{S}^{D}/\rho_0$ are shown as $v_{1J}/v_{J}^{D} = \rho_{S}^{D}/\rho_0$ in Fig. 8 at each density. Clearly as quoted for $v_{1J}/v_{J}^{D}$, all $\rho_{S}^{D}/\rho_0$ are less than one. Also from the fits $\rho_{S}^{D}/\rho_0$ is at least approximately independent of temperature over a temperature range of $0.2 \leq T \leq 2.5$ K. The $\rho_{S}^{D}/\rho_0$ are discussed further at the end of the Discussion.

Figure 9 shows the impact of disorder on the OBDM, $n(x)$. The chief impact is a lowering of the height of the long range tail of $n(x)$ at $x \geq 5$ Å. The lowering of the height of the tail in $n(x)$ may be interpreted as a modest suppression of the superfluid order by disorder. This is consistent with the modest suppression of $\rho_S/\rho_0$ by disorder seen above. The oscillations in $n(x)$ with $x$ at high density are little changed by disorder. Similarly, no oscillations in $n(x)$ are introduced by disorder at low density. Generally, the OBDM is less impacted by disorder than the $\rho_S/\rho_0$. Similarly, the pair correlation function, $g(x)$, shown in Fig. 10 is little modified by disorder.

FIG. 5. The PIMC OBDM, $n(x)$, at five linear densities between $N/L = \rho_0 = 0.117$ Å$^{-1}$ and 0.25 Å$^{-1}$. The amplitude of the oscillations in $n(x)$ is larger at high density where the liquid has solidlike correlations. The dotted lines are fits of Eq. (4) modified to take account that the periodic BCs used have a finite $L$. Only the first term of these equations was retained with $B_1$ and $K$ as adjustable parameters. The heavy dashed-dotted line at $N/L = 0.25$ Å$^{-1}$ is a fit of Eq. (4), assuming $L \rightarrow \infty$ limit. Clearly, the approximation $L \rightarrow \infty$ introduces little error. The heavy dotted line at $N/L = 0.117$ Å$^{-1}$ is a fit of the finite temperature $n(x)$ in a region of $x$ where $n(x)$ shows exponential decay.

FIG. 6. (a) The DMC pair distribution function, $g(x)$, at twelve linear densities $\rho_0$ (in Å$^{-1}$). The LL $g(x)$ fits well retaining the first term only in the series in Eq. (3) with $A_1$ and $K$ as adjustable parameters. As found for the OBDM, the amplitude of the oscillations in $g(x)$ increases with increasing density $\rho_0$. The LL parameter $K$ obtained from $g(x)$ agrees well with $K$ obtained from the equation of state and $S(q)$. (b) The PIMC pair distribution function at $T = 0.5$ K (dashed lines) and $T = 1.5$ K (full lines) at two linear densities. At sufficiently large $x$ the correlation functions at finite $T$ start to decay exponentially.

FIG. 7. The DMC static structure factor $S(q)$ versus linear density $\rho_0 = a^{-1}$ in Å$^{-1}$. At larger $\rho_0$, $S(q)$ shows a peak at $q = 2\pi/a$. At the highest $\rho_0$, there is also a small peak at the second reciprocal lattice vector $q = 4\pi/a$. The LL parameter obtained from $S(q)$ agrees well with that obtained from $e(\rho_0)$ and $g(x)$. 

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FIG. 8. PIMC superfluid fraction, $\rho_S/\rho_0$, of 1D liquid $^4$He (data points) in a nanopore (radius $R = 3$ Å) having disordered walls. Five linear densities $\rho_0$ between $\rho_0 = 0.10$ Å$^{-1}$ and $\rho_0 = 0.25$ Å$^{-1}$ with corresponding LL parameters $K > 2$ and $K < 0.5$ are shown. The solid line is the LL prediction fitted to the PIMC $\rho_S/\rho_0$ with $\hbar v_J$ a free fitting parameter. The fit is not good. The $\chi^2$ is generally large, typically more than ten times the $\chi^2$ found for the uniform case. The best fit $\hbar v_J$ differs from uniform liquid value $\hbar v_0$. When interpreted in terms of the dynamical superfluid fraction, $\rho_D/\rho_0$, defined by Machta and Guyer [54] and Prokof’ev and Svistunov [45] as discussed in the text, the ratio $v_J/v_0^0$ is interpreted as $\rho_D/\rho_0$.

V. DISCUSSION

In the previous section, we presented PIMC and DMC calculations of liquid helium confined to 1D in a narrow nanopore over a wide range of density. The Luttinger liquid parameter varied from $K \geq 2$ to $K \leq 0.5$. Both uniform nanopores and nanopores containing disorder were investigated. The goal was to test how well a 1D liquid of $^4$He atoms is described by LL theory.

A. Uniform 1D fluids

The $\rho_S/\rho_0$ of 1D liquid $^4$He in a uniform nanopore was accurately described by LL theory. The LL expression, Eq. (7),
The DMC OBDM, \( n(x) \), equally shows oscillations in its long-range tail unique to 1D. Both the PIMC \( g(x) \) and \( n(x) \) were well fitted by the LL expressions, Eqs. (3) and (4), respectively, for distances smaller than the thermal length. For high enough temperatures and long lengths, exponential decay of the correlation functions was observed. Near the spinodal density, \( g(x) \) appeared to depend on temperature and to fall below unity, which is attributed to close proximity to droplet formation. There were no significant departures from LL predictions over the range of densities and \( K \) values investigated.

For a LL the height of the sharp peak in \( S(q) \) at \( q = 2\pi r_0 \) is predicted to scale with the number of bosons in the sample \( N \) as \( S(2\pi r_0) = AN^{1-2K} \). We tested this scaling at linear density \( \rho_0 = 0.25 \, \text{Å}^{-1} \) and found that \( S(2\pi r_0) \) does indeed scale as \( N^{1-2K} \) and the best fit \( K = 0.344(14) \) obtained from the scaling agrees well with \( K \) calculated by other methods (see Fig. 3). The scaling suggests that in the thermodynamic limit the peak height is infinite for \( K < 1/2 \). For density where \( K > 1/2 \) no increase of the main peak with the system size is observed, suggesting a finite value of the peak in the thermodynamic limit.

Liquid parahydrogen (p-H\(_2\)) in 1D [48,49] or quasi-1D (e.g., at the center of carbon nanotubes or in harmonic potential confinement) [24,49–51,59] is similar to the present 1D \(^4\)He liquid. The aim in confining p-H\(_2\) in porous media is to suppress the freezing temperature of the liquid so that temperatures low enough to observe superfluidity might be reached. Recent reviews of this literature appear in Refs. [51,59]. The aim of investigating 1D is to discover 1D superfluidity and to compare

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**FIG. 9.** As Fig. 8 for the PIMC OBDM \( n(x) \) at three densities \( \rho_0 \); solid lines in disorder, dashed lines in uniform pores. Disorder decreases the OBDM at longer \( x \). In contrast to \( \rho_5/\rho_0 \), the LL prediction for \( n(x) \), shown in Fig. 5, fits the PIMC OBDM in disorder quite well, especially at higher densities where the amplitude of the oscillations in \( n(x) \) are observable. At \( \rho_0 = 0.1 \, \text{Å}^{-1} \), the heavy dashed lines show a fit of the finite temperature \( n(x) \) (e.g., Eq. (71) of Ref. [58]), \( n(x) = \rho_0 B_0(\pi/L_T)/\rho_0 \sinh(\pi x/L_T) \) where \( L_T = \pi^2\rho_0/\rho_0 (\pi = h^2/mk_B) \) in the region of \( x \geq L_T \) where \( n(x) \) decays exponentially with \( x \). At shorter \( x \), the finite \( T \) \( n(x) \) reduces to the \( T = 0 \) K result, Eq. (4).

**FIG. 10.** As Fig. 8 for the DMC pair distribution function \( g(x) \).
with LL theory. In these investigations, it is assumed that superflow in 1D liquid p-H$_2$ is possible if (and only if) the LL parameter $K$ exceeds 3/2, a criterion predicted by LL theory [52].

At high linear density, $S(q)$ of 1D liquid p-H$_2$ shows a large peak at $q = 2\pi \rho_0$ with corresponding large amplitude oscillations in $g(x)$ characteristic of solidlike atomic correlations, as found here for 1D liquid $^4$He at high density. At high $\rho_0$, the LL parameter is small, $K \leq 0.5$. However, when the density is reduced $K$ increases, as found here for 1D $^4$He. When $K \geq 3/2$ at low density ($\rho_0 \lessapprox 0.75$ Å$^{-1}$) has been reported [50]. However, for 1D and quasi-1D p-H$_2$, the spinodal density is reported [59,60] at $\rho_0 = 0.20$ Å$^{-1}$. This means the low density where $K > 3/2$ is reported [50] cannot be reached.

Near the equilibrium density [48,49,59], $\rho_0 = 0.22$ Å$^{-1}$ ($K \approx 0.35$), the $S(q)$ and $g(x)$ of 1D p-H$_2$ are very similar to those found here for 1D $^4$He at $\rho_0 = 0.22$ Å$^{-1}$. The two 1D liquids appear to be quite similar except that a much lower density can be reached in 1D liquid $^4$He before the liquid becomes unstable. $K$ always diverges at the spinodal density, but the density range over which $K$ is large can be very small.

Bertaina et al. [61] have evaluated the LL parameter $K$ and the dynamic structure factor of pure 1D $^4$He. At high density, $\rho_0 > 0.2$ Å$^{-1}$, the $K$ of pure 1D $^4$He and $^4$He in the present nanopore (Fig. 3) are the same. At lower density, $K$ is somewhat lower for pure 1D $^4$He since the spinodal density is lower in pure 1D $^4$He ($\rho_{p} = 0.026$ Å$^{-1}$).

Similarities can also be found between the present system and pure 1D $^3$He [62] and spin-polarized hydrogen isotopes [63], especially at high densities in the quasisolid regime where $K < 1/2$. Due to hard-core interactions the energy and diagonal properties do not depend on the statistics but rather on the mass. At high linear densities the LL parameter $K$ as a function of density is almost the same for all three hydrogen isotope systems [63]. $^3$He enters [62] the $K < 1/2$ regime at a linear density of 0.19 Å$^{-1}$ which is quite close to the value obtained in this work. The difference arising from the mass becomes more apparent at lower density. In particular, neither of the 1D systems mentioned possesses a two-body bound state. Thus as the density is lowered the $K$ parameter either increases up to 1 (for spin-polarized hydrogen) or shows a nonmonotonic behavior, reaching a maximum higher than 1 and decreasing towards 1 as the density goes to zero. The height of the maximum increases in the vicinity of the two-body bound states.

In the present model, disorder is introduced by adding impurities at random locations in the pore walls near the liquid. There can be a small variation in the magnitude of the disorder with periodic length $L$, an end effect. This variation with $L$ would contribute to the departure of $\rho_S/\rho_0$ from LL scaling, i.e., to $\chi^2$. We performed some exploratory calculations by creating disorder configurations at larger $L$ as multiples of those from smaller ones, in particular $L = 60$ Å as a multiple of $L = 30$ Å or by using different random configurations. That had a small effect on the quality of fits ($\chi^2$ changed from 16 to 20) and thus could not account for the increase of the $\chi^2$ from the homogeneous case, in particular for the highest densities, where the largest $\chi^2$ were obtained. From the pair distribution function at lower densities and creation of faint aperiodic oscillations, it appears that particles are affected by the creation of random pockets of attraction and repulsion. The effect is larger for shorter lengths and may explain in part the increase of the $\chi^2$ with respect to the homogeneous case. In summary, since large values of $\chi^2$ were found at high $\rho_0$ and systematic departures from LL scaling at long $L$ were obtained, we do not think the end effect makes a significant contribution to $\chi^2$ and departures from LL character.

LL theory [52] also predicts 1D superfluidity to be fragile in disorder for $K < 3/2$, robust to disorder for $K > 3/2$. We found $\rho_S/\rho_0$ to be robust at $K > 3/2$. The fragility of $\rho_S/\rho_0$ in disorder at $K < 3/2$ was observed only in the long $L$ samples. That is, at $\rho_0 > 0.2$ Å$^{-1}$ where $K < 3/2$, $\rho_S/\rho_0$ was suppressed in disorder at long $L$ only, e.g., at $\rho_S/\rho_0 = 0.2$ Å$^{-1}$ and $L = 90$ Å, $\rho_S/\rho_0$ is nearly zero at all $T$ investigated. Since the prediction draws on renormalization group methods, a long sample may be required to observe the suppression by disorder. Similarly, the OBDM was reduced by disorder more at long lengths ($x$) than at short lengths in Fig. 9.

At lower densities, where $K > 3/2$ and we expect the superfluidity to be robust, we have examined one additional, somewhat simpler model. Fixed fictitious particles of linear density 0.1 Å$^{-1}$ were added randomly along the length and all angles, at a distance 2.5 Å from the axis. Half of the particles were interacting attractively and half repulsively with the He atoms via the potential $\pm D/r^3$, with $D = 10$ K Å$^3$. The average of this additional interaction potential is zero, and its maximum within the space occupied by the helium atoms was less than 10% of the potential energy per particle. Although the values of the superfluid fraction were less suppressed below the uniform case, Luttinger liquid scaling with $LT$ was not observed. Again, superfluidity was suppressed more at long lengths than would be expected from the suppression at shorter lengths.

The impact of disorder on superfluidity is predicted to depend on the strength of the disorder [38]. It is thus possible that by reducing $D$ or $\epsilon_d$ one could recover the LL scaling in disorder. Our results, however, suggest, that LL scaling would be preserved only for very weak disorder. The additional difficulty with $^4$He is the proximity to the spinodal decomposition, which prevents access to lower densities where $K$ would be larger.

In an extensive study of hard core bosons on a lattice with weak on site disorder, Doggen et al. [64] find a critical value
of $K_c = 3/2$ in agreement with the predictions of Giamarchi and Schultz [52] (GS). The disorder is white noise in the sense that the disorder on nearest neighbor sites is uncorrelated. This agreement suggests that although the prediction of GS was derived using weak Gaussian correlated disorder the prediction may be valid for weak disorder more generally. For strong disorder, Doggen et al. find $K_c > 3/2$ and $K_c$ is no longer constant. Rather $K_c$ depends on the magnitude of the disorder.

Again, our findings are limited. At low linear density $\rho_0$ where $K > 3/2$ we find $\rho_S/\rho_0$ is not suppressed by disorder. However, at higher density $\rho_S/\rho_0 = 0.2 \, \text{Å}^{-1}$ where $K = 0.7$ we find $\rho_S/\rho_0$ is suppressed to near zero for the long $L$ samples consistent with GS predictions.

We may also interpret the PIMC $\rho_S/\rho_0$ as an opportunity to determine the dynamical $\rho_D^0/\rho_0$ proposed in Refs. [45,54]. In this case, the fits to $\rho_S/\rho_0$ in Fig. 8 are interpreted as fits of Eq. (16) to the PIMC $\rho_S/\rho_0$ with $\rho_D^0/\rho_0$ in Eq. (15) as the fitting parameter. Before discussing the resulting $\rho_D^0/\rho_0$, we compare some expressions for $\rho_S/\rho_0$ depending on the fluid and disorder, $vJ > v^0$ or $vJ < v^0$.

The present paper, Eq. (7) was fitted to the PIMC $\rho_S/\rho_0$ in terms of the “dynamical” superfluid fraction $\rho_D^0/\rho_0$ introduced in Sec. II,

$$\frac{\rho_S}{\rho_0} = \rho_D^0/\rho_0 \left[ 1 - 4\pi^2 \left( \frac{J^2}{\alpha} \right) \right],$$

where $\alpha = \alpha_0(v_J/v^0_J)^{-1}$, $\alpha_0 = (TL/\sigma \rho_0)$ as before. If the LL Hamiltonian, Eq. (1), describes the liquid, the probability of observing a winding number $T$ across the liquid of length $L$ is [46] $P(T) \propto e^{-\langle 2\pi^2 \alpha J^2 \rangle}$. Since $\langle J^2 \rangle \rightarrow 0$ as $T \rightarrow 0$, Eq. (20) gives $\rho_S/\rho_0 \rightarrow 1$ at $T = 0$ K. Equations (6) and (20) are similar except Eq. (6) gives $\rho_S/\rho_0 \rightarrow v_J/v^0_J$ at $T = 0$ K, while Eq. (20) gives $\rho_S/\rho_0 \rightarrow 1$. The two expressions are identical for $v_J = v^0_J$. In earlier publications [20,22,43], Eq. (20) was fitted to PIMC values of $\rho_S/\rho_0$.

Prokoﬁev and Svistunov [45] have derived an expression for $\rho_S/\rho_0$ in terms of the “dynamical” superfluid fraction $\rho_D^0/\rho_0$ introduced in Sec. II,

$$\frac{\rho_S}{\rho_0} = \rho_D^0/\rho_0 \left[ 1 - 4\pi^2 \left( \frac{J^2}{\alpha^D} \right) \right],$$

where $\alpha^D = \alpha_0(\rho_D^0/\rho_0)^{-1}$ as in Eq. (15). Equations (16) and (21) are identical. Specifically, the $T = 0$ limits of Eqs. (16) and (21) are $\rho_S/\rho_0 = \rho_D^0/\rho_0$. Although the physical meaning of Eqs. (16) and (21) is very different from that of Eq. (6), the equations are identical with $\rho_S/\rho_0$ playing the role of $v_J/v^0_J$.

Also, from Eq. (21) we see clearly that since PIMC values of $\rho_S/\rho_0$ satisfy $\rho_S/\rho_0 \leq 1$, a fit of Eq. (16) or Eq. (21) to PIMC $\rho_S/\rho_0$ will always give $\rho_D^0/\rho_0 \leq 1$.

### 2. Dynamical superfluid fraction

From Eq. (21), we see clearly that the “dynamical” $\rho_D^0/\rho_0$ and the “thermodynamic” $\rho_S/\rho_0$ are the same at $T = 0$ K. The central question is then, what is the temperature dependence of $\rho_D^0/\rho_0$? For a uniform (Galilean invariant) 1D liquid that is well described by the LL Hamiltonian with $v_J = v^0_J$, given the identity of Eqs. (7) and (16), we have $\rho_D^0/\rho_0 = 1$ for a uniform 1D liquid at all temperatures. Similarly, for a 1D liquid in disorder, given the fits shown in Fig. 8, there is no apparent temperature dependence in $\rho_D^0/\rho_0$ needed to get a reasonable fit of Eq. (16) to the PIMC $\rho_S/\rho_0$. For example, at $\rho_0 = 0.2 \, \text{Å}^{-1}$ the fit is made over a range of $LT$ values given by $0.5 < (L/\hbar v^0_J) < 5$, where $L/\hbar v^0_J = (LT/\pi \sigma_0)$. It is difficult to translate this into a definite temperature range, but a range $0 < T < 2.5$ K was used in calculations. This suggests a $\rho_D^0/\rho_0$ roughly independent of $T$ up to $T \sim 2$ K. A $\rho_D^0/\rho_0 \approx 0.85 \pm 0.15$ independent of temperature is suggested at the considered densities.

The “dynamical” $\rho_D^0/\rho_0$ has been evaluated for atomic gases in a periodic external potential [27]. Both $\rho_D^0/\rho_0$ and $\rho_S/\rho_0$ are found to go from unity to zero as the density goes from incommensurate to commensurate, a superfluid to Mott insulator transition.

### C. Comparison with experiment

The thermodynamic $\rho_S/\rho_0$ given by Eq. (5) and evaluated using PIMC has been enormously successful in reproducing the observed $\rho_S/\rho_0$ in 2D and 3D fluids [4,6,53,65]. It is tempting to compare the present model and thermodynamic $\rho_S/\rho_0$ and OBDM with measurements of liquid $^4$He in fully filled nanopores [14,15,66] of diameter 28 Å and 47 Å. However, the $\rho_S/\rho_0$ observed in these nanopores has a transition temperature (e.g., $T_c = 0.9$ K for $d = 28$ Å) characteristic of 2D or 3D. In contrast, the $\rho_S/\rho_0$ in 1D has no thermal $T_c$. The temperature dependence of $\rho_S/\rho_0$ observed in 28 Å nanopores [14] below $T_c$ is also quite different from that of $\rho_S/\rho_0$ in 1D.

In larger nanopores, the phase at temperatures above $T_c$ is a Bose glass (BG) phase [18,19,67,68]. The BG phase supports well defined phonon modes in puddles of BEC separated by normal fluid [66,68,69]. The interpretation of the BG phase in terms of puddles of BEC has recently been made in Bose gases in traps [70].

Rather than 1D like, the observed $\rho_S/\rho_0$ in nanopores [14] looks 2D like with a $T_c$. In nanopores of diameter 28 Å (liquid pore radius $R \approx 9$ Å), the liquid $^4$He is deposited predominantly in 2D layers [20,43]. A PIMC calculation of $\rho_S/\rho_0$ and the OBDM, $n(x)$, of $^4$He in the liquid layers at $R = 9$ Å [43] shows that both $\rho_S/\rho_0$ and $n(x)$ are 2D like [43] with a $T_c = 1.4$ K, which is close to the $T_c = 0.9$ K observed in nanopores, $d = 28$ Å. Extensive measurements of $\rho_S/\rho_0$ in films of $^4$He in nanopores have also been made [11,13]. In interconnected
VI. CONCLUSION

The present DMC and PIMC results show that homogeneous liquid $^4$He confined to 1D in a uniform nanopore is well described by LL theory. When disorder is added to the same nanopore, the PIMC superfluid fraction $\rhoS/\rhoD$ retains only roughly the shape expected for 1D. The $\rhoS/\rhoD$ does not scale well with $LT$ and a fit of the LL expression has a large $\chi^2$. In disorder the $\rhoS/\rho0$ becomes length dependent and $\rhoS/\rho0$ is suppressed to near zero by disorder at $K < 3/2$ only in the longer length samples. The long range part of the OBDM is well described by LL theory. When disorder is added to the same nanopore, the PIMC superfluid fraction $\rhoS/\rho0$ is calculated in terms of the thermodynamic, PIMC $\rhoS/\rho0$ in disorder. The two are the same at $T = 0$ K. Within precision we find $\rhoS^2/\rho0$ is independent of temperature (e.g., up to 2 K in 1D liquid $^4$He).

ACKNOWLEDGMENTS

The authors gratefully acknowledge Massimo Boninsegni for providing the PIMC code used in this work and for valuable discussions. They thank Ian Affleck, Adrian Del Maestro, and Marcus Holtzmann for valuable discussions and insights on reduced dimensional systems and phase transitions. H.R.G. thanks the Theory Group, Institut Laue Langevin for hospitality where part of this paper was written. This work was supported by the US DOE, Office of Basic Energy Sciences (BES), Grant No. DEFG02-03ER46038 and in part by the Croatian Science Foundation under the project number IP-2014-09-2452 and Croatian Ministry of Science and Education.