Dynamics of quantum liquids at high momentum

B. Tanatar, E. F. Talbot, and H. R. Glyde

Department of Physics, University of Delaware, Newark, Delaware 19716

(Received 19 June 1987)

The dynamic form factor $S(Q,\omega)$ in liquid $^3$He and $^4$He is evaluated in the wave-vector transfer range $3 \leq Q \leq 15 \ \text{Å}^{-1}$. The input is the pair interatomic potential, developed by Aziz et al. The $S(Q,\omega)$ is calculated within the random-phase approximation (RPA) which becomes valid when $\hbar Q$ is much larger than the average momentum in the liquid. A T-matrix approximation represents the interaction appearing in the RPA. The aim is to explore how well $S(Q,\omega)$ can be described for $3 \leq Q \leq 15 \ \text{Å}^{-1}$ from first principles. In $^3$He, we find $S(Q,\omega)$ is a broad, nearly Gaussian function, centered just below the recoil frequency having a width and shape that agrees well with experiment. It does, however, have tails at high frequency which make important contributions to its moments. In $^4$He, $S(Q,\omega)$ is a more sharply peaked function which also agrees quite well with experiment. We are able to reproduce the oscillations in the peak position and in the width of $S(Q,\omega)$ with $Q$ in liquid $^4$He observed by Martel et al. In the present model, these oscillations originate from oscillations in the magnitude of the T-matrix interaction with $Q$. The corresponding oscillations are predicted to be very small and probably unobservable in liquid $^3$He.

I. INTRODUCTION

The study of excitations in liquid $^4$He by inelastic neutron scattering has a long and rich history.1–7 Observed in these experiments is the coherent dynamic form factor, $S_c(Q,\omega)$, where $\hbar Q$ and $\hbar \omega$ are the momentum and energy transferred from the neutron to the fluid, respectively. For a liquid, $S(Q,\omega)$ is a function of $|Q| = Q$ only. At low $Q$, $S(Q,\omega)$ is dominated by a single peak characteristic of scattering in which the neutron creates a well-defined collective excitation, the renowned phonon-roton mode proposed by Landau in $^4$He. At high $Q$ ($Q \geq 3 \ \text{Å}^{-1}$) this mode disappears and $S(Q,\omega)$ is a broad function of $\omega$, characteristic of scattering from weakly interacting single particles.8 In the range $3 \leq Q < 10 \ \text{Å}^{-1}$, Cowley and Woods8 and Martel et al.9 observed that the peak positions and the width of $S(Q,\omega)$ oscillates with $Q$. More precisely, the full width at half maximum, $W(Q)$, oscillates with $Q$ around a mean value $W(Q)/Q = 19\ \text{K Å}$, having a period of approximately $3 \ \text{Å}^{-1}$ and a first maximum at $Q \approx 4 \ \text{Å}^{-1}$. In a model inspired by the work of Hohenberg and Platzman,10 Martel et al.9 related the oscillations in $W(Q)$ to oscillations in the He-He atom cross section. The scattering cross section, $\sigma(Q)$, for two He atoms scattering in free space is indeed observed11 to oscillate with $Q$, the so-called glories.

Studies of excitations in liquid $^3$He by inelastic neutron scattering were first reported12,13 in 1974. Observed in $^3$He is a sum of $S_c(Q,\omega)$ and a spin-dependent dynamic form factor, $S_f(Q,\omega)$.

\[ S(Q,\omega) = S_c(Q,\omega) + \frac{\sigma_i}{\sigma_e} S_f(Q,\omega), \]

where $\sigma_i/\sigma_e$ is the ratio of the incoherent bound-atom cross sections. At $Q \leq 1.5 \ \text{Å}^{-1}$, the density excitations observed in $S_c(Q,\omega)$ respond collectively in a zero-sound mode.13–15 The spin-density component $S_f(Q,\omega)$ has a sharp resonance at low energy identified as a paramagnon resonance.14 For $Q \gtrsim 1.5 \ \text{Å}^{-1}$, the sharp components of $S(Q,\omega)$ disappear and the total $S(Q,\omega)$ is a broad function13–16 of $\omega$ characteristic of scattering from weakly interacting particle-hole (p-h) excitations. An interesting question is whether the width of $S(Q,\omega)$ in $^3$He also oscillates with $Q$ in the range $3 \leq Q < 10 \ \text{Å}^{-1}$, due to $^3$He–$^3$He atom interactions. Sokol et al.17 have recently observed $S(Q,\omega)$ at $12 \leq Q \leq 15 \ \text{Å}^{-1}$ in $^3$He and found a width $W(Q)/Q = 1.9 \pm 0.3 \ \text{meV Å} = 22 \pm 3 \ \text{K Å}$, independent of $Q$. Mook18 has observed $S(Q,\omega)$ in the range $3 \leq Q \leq 7 \ \text{Å}^{-1}$. He found an average width of $W(Q)/Q = 2.18 \ \text{meV Å}$, one which has a minimum at $Q \approx 5.5 \ \text{Å}^{-1}$ increases significantly between 5.5 and 7 Å⁻¹, but has no actual oscillations in $W(Q)/Q$. Assuming $S(Q,\omega)$ has a Gaussian shape, Sokol et al.17 and Mook18 evaluated the second moment, $M_2$, of their observed $S(Q,\omega)$ using the relation $\hbar^2 M_2 = W/8 \ln 2$ valid for Gaussian functions. The atomic kinetic energy $\langle E_K \rangle$ is related to the second moment, $M_2$, of the incoherent $S_c(Q,\omega)$ by $\hbar^2 M_2 = \frac{1}{2} \langle E_K \rangle \hbar \omega_K$ where $\omega_K$ is the recoil frequency. Assuming $S(Q,\omega)$ is well approximated by $S_c(Q,\omega)$ at these high-$Q$ values ($M_2 \approx M_c$), Sokol et al. and Mook obtained values of the kinetic energy $\langle E_K \rangle = 8.1 \pm 1.3$ and $\langle E_K \rangle = 10.7 \ \text{K}$, respectively, in $^3$He. These lie below the most recent evaluations19–22 giving $\langle E_K \rangle \approx 13 \ \text{K}$.

The purpose of the present paper is to evaluate $S(Q,\omega)$ in liquid $^3$He and $^4$He in the range $3 \leq Q \leq 15 \ \text{Å}^{-1}$. The aim is to explore the shape of $S(Q,\omega)$ and particularly its width. The calculations are first-principle ones, using as input only the pair interatomic potential and the observed density of the fluid. Can the observed oscillations in the $W(Q)/Q$ of $S(Q,\omega)$
in liquid $^4$He be reproduced? Are oscillations in $W(Q)/Q$ in liquid $^4$He expected? Is the shape of $S(Q,\omega)$ predominantly Gaussian, or does it tails reaching up to high frequency, as has been predicted and anticipated\textsuperscript{23} from many points of view? Our goal is an assessment of interatomic interactions and their contribution to $S(Q,\omega)$. Particularly, we are not here assessing the validity of the impulse approximation\textsuperscript{10,24} (IA) to $S(Q,\omega)$. The IA may be used at high $Q$ to obtain the momentum distribution, $n_p$, giving the fraction of atoms having momentum $p$. Of special interest\textsuperscript{10,12,15} is the fraction of atoms in the zero-momentum state, $n_0$, in liquid $^4$He. Rather our goal is to see how well $S(Q,\omega)$, its shape, and its width, can be determined from first principles in the range $3 \leq Q \leq 15 \text{ Å}^{-1}$ where interatomic interactions are expected to be important. Specifically, in $^4$He we will choose a temperature $T$ above the critical temperature for Bose condensation where $n_0=0$.

Since liquids $^3$He and $^4$He do not respond collectively for $Q \geq 3 \text{ Å}^{-1}$, we expect interatomic correlations between pairs of atoms to be the dominant interaction at high $Q$. Here we evaluate this interaction within a $T$-matrix approximation. The $S(Q,\omega)$ is then calculated in the random-phase approximation (RPA). The specific form of the dynamic form factor in $^4$He and $^3$He and the RPA are set out in Sec. II. The reasons the full dynamic susceptibility can be well approximated by the RPA at high $Q$ are set out in Sec. III. Models for the $T$ matrix and the zero-order dynamic susceptibility are presented in Sec. IV. The results for $S(Q,\omega)$ are displayed in Sec. V with a discussion in Sec. VI.

### II. DYNAMIC FORM FACTOR

Since $^4$He is a spinless boson and liquid $^4$He is extremely isotopically pure (all nuclei identical), the neutron inelastic scattering cross section is entirely coherent and proportional to the dynamic form factor\textsuperscript{26}

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \frac{1}{N} \langle \rho(Q,t)\rho(-Q,0) \rangle . \tag{2}$$

Here

$$S_{c,\ell}(Q,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \frac{1}{N} \langle \rho_{\ell}(Q,t)\rho_{\ell}(-Q,0) \rangle$$

$$= 2[S_{\downarrow\downarrow}(Q,\omega) \pm S_{\uparrow\uparrow}(Q,\omega)], \tag{8}$$

where the plus sign refers to $S_{\uparrow\uparrow}$. Thus neutron studies of $^3$He observe correlations in both the number and the spin density. The ratio of the incoherent to coherent bound-atom cross sections is believed\textsuperscript{14} to be $\sigma_{\downarrow}/\sigma_{\uparrow} = 0.25$, the value we use here. At high $Q$ ($Q \geq 3 \text{ Å}^{-1}$), $S_{\uparrow\uparrow}(Q) = S_{\downarrow\downarrow}(Q) \approx 1$ so that approximately 75\% of the scattering is coherent.

If there are no correlations in the spins, $\langle I_{\ell} I_{\ell} \rangle = I(I+1)\delta_{\ell\ell}$, the $S_{\ell\ell}$ in (4) reduces precisely to the usual incoherent form factor,\textsuperscript{26}

$$\rho(Q,t) = \sum_{k} e^{-iQ\cdot r_{k}(t)} = \sum_{k} a^\dagger_k(t)a_{k+Q}(t) \tag{3}$$

is the Fourier transform of the number density $\rho(r,t)$. Thus neutron studies observe correlations in the $^3$He atom number density.

Since $^3$He and neutrons are spin-$\frac{1}{2}$ fermions, a neutron-$^3$He nucleus pair interact in either a relative spin-triplet or spin-singlet state. Thus the inelastic scattering cross section for neutrons from liquid $^3$He depends on the spin orientations of the $^3$He nuclei. For an unpolarized beam or sample, it is proportional\textsuperscript{27,28} to the dynamic form factor in (1) where $S_c$ is given by (2) and

$$S_c(Q,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \frac{1}{N} \langle I(Q,t)I(-Q,0) \rangle \tag{4}$$

is the spin-dependent part. The

$$I(Q,t) = \sum_{l} I_l(t)e^{-iQ\cdot r_l(t)} = \frac{1}{2} \sum_{k} a^\dagger_k(t)\hat{\sigma}a_{k+Q}(t) \tag{5}$$

is the Fourier transform of the spin density $I(r,t) = \sum_{l} I_l(r-r_l(t))$,

and $\hat{\sigma}$ is the Pauli spin matrix. For $^3$He $\rho(Q,t)$ in (2) is a sum over both spin states,

$$\rho(Q) = \sum_{k,\lambda} a^\dagger_{k\lambda}a_{k+Q\lambda} = \rho_{\uparrow}(Q) + \rho_{\downarrow}(Q) . \tag{6}$$

Also, since the fluid is isotropic, we may write

$$\langle I(Q)I(Q) \rangle = \frac{3}{2} \langle I_1(Q)I_{1\downarrow}(Q) \rangle$$

$$= \frac{3}{2} \langle [\rho_{\uparrow}(Q) - \rho_{\downarrow}(Q)]^2 \rangle . \tag{7}$$

The last equality follows from (5) using the $z$ component $\hat{\sigma}_z$ of the Pauli matrix. With $I(I+1) = \frac{1}{2}$ the coherent and spin-dependent dynamic form factors in $^3$He may be written as

$$\rho(Q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t}\langle e^{-iQ\cdot r_{1\downarrow}(t)}e^{iQ\cdot r_{1\uparrow}(0)} \rangle \tag{9}$$

which is the Fourier transform of the self-correlation function $S_1(Q,t)$ of one of the atoms $l$. In this case the static form factor is precisely

$$S_1(Q) = \int d\omega \, S_1(Q,\omega) = S_1(Q,t=0) = 1 . \tag{10}$$

The $S_1(Q)$ observed using neutrons\textsuperscript{2} and calculated by Monte Carlo methods\textsuperscript{29} is $S_1(Q) \approx 1$ for $Q \geq 0.8 \text{ Å}^{-1}$. 

Thus correlations between spins appear to be weak and \( S_i \) is probably very well approximated by \( S_i \) for \( Q \geq 3 \quad \text{\AA}^{-1} \). Here we use (2) for \(^4\text{He} \) and (1) and (8) for \(^3\text{He}\) to evaluate \( S(Q, \omega) \).

We introduce a dynamic susceptibility corresponding to \( S_i \) and \( S_j \),

\[
\chi_{c, I}(Q, t) = -\frac{i}{\Omega} \left\langle T \left[ \rho_i(Q, t) \pm \rho_i(Q, t) \right] \right\rangle_x (Q, 0) \pm \rho_i(Q, 0))
\]

where \( \Omega \) is the volume and \( T \) is the time-ordering operator. The Fourier transform of \( \chi_{c, I}(Q, t) \) is

\[
\chi_{c, I}(Q, \omega) = \int dt e^{i\omega t} \chi_{c, I}(Q, t)
\]

This is related to \( S(Q, \omega) \) by

\[
S(Q, \omega) = -\frac{1}{n} \chi_{c, I}(Q, \omega)
\]

at \( T=0 \) K where \( n = N/\Omega \) is the number density. In Sec. III we discuss the general equation for \( \chi(Q, \omega) \) in \(^3\text{He} \). We argue that for \( Q \geq 3 \quad \text{\AA}^{-1} \) this equation may be well approximated by the RPA result, \( ^{30,31} \)

\[
\chi_{c, I}(Q, \omega) = \frac{\chi_{c, I}(Q, \omega)}{1 - I_{s, a}(Q, \omega) \chi_{c, I}(Q, \omega)}
\]

Here

\[
\chi_{c, I}(Q, \omega) = \frac{2\hbar}{\Omega} \sum_{\mathbf{1}} \left[ \frac{n_i(1-n_i+Q)}{d_1(\omega)+i\eta} - \frac{n_i(1-n_i)}{d_1(-\omega)-i\eta} \right]
\]

is the Lindhard function \(^{32} (1 \equiv p_1) \). The \( \chi_{c, I}(Q, \omega) \) describes the response of a system of noninteracting fermions and

\[
d_1(\omega) = \hbar\omega - \epsilon_i + Q(\epsilon_i + \omega) + \epsilon_i,
\]

\[
d_1'(\omega) = \hbar\omega - \epsilon_i + Q + \epsilon_i(\epsilon_i + Q - \omega).
\]

The \( \epsilon_i + Q(\epsilon_i + \omega) \) is the single-particle energy of particle having momentum \( p_i + q \) at energy \( \epsilon_i + \omega \). The \( \epsilon_i = \epsilon_i(\epsilon_i) \) is the on-shell single-particle energy. In general the \( \epsilon_i \) are the full or renormalized single-particle energies due to interactions in the fluid. The \( \chi_{c, I}(Q, \omega) \) is clearly a sum of two terms which we may write as

\[
\chi_{c, I}(Q, \omega) = \chi_i(Q, \omega) + \chi_2(Q, \omega)
\]

By replacing \( 1 + Q \rightarrow -1 \) in the \( \chi_2 \) term in (14) we have

\[
\chi_{c, I}(Q, \omega) = \frac{2\hbar}{\Omega} \sum_{\mathbf{1}} \left[ \frac{n_i(1-n_i+Q)}{d_1(\omega)+i\eta} + \frac{n_i(1-n_i+Q)}{d_1(-\omega)+i\eta} \right]
\]

\[
= \chi_i(Q, \omega) + \chi_3(Q, Q_+) - \chi_i(Q, -\omega).
\]

For \( Q \geq 2p_F \approx 1.6 \quad \text{\AA}^{-1} \), which we consider here, \( n_i n_i + Q = 0 \). Also, we are interested only in positive values of \( \omega \). In this case \( \chi_i(Q, -\omega) \approx 0 \) (since \( \epsilon_i + Q - \epsilon_i > 0 \)) and in the region of \( \omega \approx \omega_R \), \( \chi_i(Q, Q_+) \approx \chi_i(Q, -\omega) \). This is discussed further in the Appendix. Thus at high \( Q \) and in the region \( \omega \approx \omega_R \),

\[
\chi_{c, I}(Q, \omega) \approx \chi_i(Q, \omega) = \frac{2\hbar}{\Omega} \sum_{\mathbf{1}} \frac{n_i}{d_1(\omega)+i\eta}.
\]

The interaction \( I(Q, \omega) \) has a spin-symmetric component

\[
2I_i(Q, \omega) = I_{1,11}(Q, \omega) + I_{1,11}(Q, \omega)
\]

which enters \( \chi_i \) and a spin-antisymmetric component

\[
2I_i(Q, \omega) = I_{1,11}(Q, \omega) - I_{1,11}(Q, \omega)
\]

which enters \( \chi_i \). We derive \( I(Q, \omega) \) in detail in the Appendix. In general, \( I(Q, \omega) \) describes the interaction between a \( p-h \) pair including all processes except those which have a single \( p-h \) pair as an intermediate state. The \( I(Q, \omega) \) may be equally well expressed as a particle-particle interaction. For \( Q \geq 1.7 \quad \text{\AA}^{-1} \) in \(^3\text{He} \), collective excitations are not supported in the fluid. At high \( Q \) we assume therefore that \( I(Q, \omega) \) is dominated by the interaction between pairs via the interatomic potential \( v(r) \) [chiefly by the steeply repulsive short range part of \( v(r) \)] and that the interaction induced by the collective excitations is weak or negligible.

### III. DYNAMIC SUSCEPTIBILITY

In this section we discuss the dynamic susceptibility \( \chi(Q, \omega) \) and show how it may be reduced to the RPA form (13) at high \( Q \). We begin with the exact equation for \( \chi \) and make the reduction by selecting approximations to the \( p-h \) interaction which should be valid at high \( Q \). We consider \( \chi_i \) since we know that the same development follows for \( \chi_i \) with \( I_i \) replaced by \( I_{ij} \).

With the aid of (13), \( \chi(Q, t) \) in (11) may be written as

\[
\chi(Q, t) = \frac{i}{\Omega} \sum_{\mathbf{1}, 2} \left\langle T \rho_1(Q, t) \rho_2^+(Q, 0) \right\rangle
\]

\[
= \frac{1}{\Omega^2} \sum_{\mathbf{1}, 2} \chi(p_1, p_2; Q, t).
\]

Here \( p_2^+(Q) = a_2^+ Q a_2 \) creates a single-particle \( (2+Q) \)-hole \( (2) \) pair. The two-point function \( \chi(p_1, p_2; Q, t) \) describes the propagation of a specific \( p-h \) pair \( (2+Q, 2) \), created at \( t=0 \), through the fluid until the pair is annihilated from state \( (1+Q, 1) \) at time \( t \). The general equation for the Fourier transform \( \chi(12, Q, \omega) \) of \( \chi(p_1, p_2; Q, t) \) is

\[
\chi(12, Q) = (2\pi)^4 \chi_{c, I}(1, Q) \delta_{1, 2}
\]

\[
+ \chi_{c, I}(1, Q) \int d3 \text{ I}(1 + Q, 3; 1, 3 + Q) \chi(12, Q).
\]

Here we use four-vector notation:

\[
1 = (p_1, \omega_1), \quad Q = (Q, \omega), \quad \delta_{1, 2} = \delta_{p_1, p_2} \delta_{\omega_1, \omega_2},
\]

and

\[
\chi_{c, I}(Q, \omega) \approx \chi_i(Q, \omega) = \frac{2\hbar}{\Omega} \sum_{\mathbf{1}} \frac{n_i}{d_1(\omega)+i\eta}.
\]
The interaction between pairs via the repulsive core of \( v(r) \) can be described by a \( T \) matrix. The \( T \) matrix describes the interaction via the whole of \( v(r) \) but is particularly accurate for the steeply repulsive region. Thus we assume

\[
I(12;34) = \Gamma(12;34)
\]

where \( \Gamma \) is the \( T \) matrix. For two particles scattering in free space or in fluid we will see in Sec. III and in the Appendix that

\[
\Gamma(1+Q,3;1,3+Q) = \Gamma(k,k';P,E).
\]

That is, the \( T \) matrix depends upon

\[
k = \frac{1}{2} \left[ (p_1+Q) - p_3 \right], \text{ relative incoming wave vector}
\]

\[
k' = \frac{1}{2} \left[ p_1 - (p_3+Q) \right], \text{ relative outgoing wave vector}
\]

\( P = p_1 + p_3 + Q \), c. m. momentum

\[
E = \omega_1 + \omega_3 + \omega, \text{ total energy}.
\]

For particles scattering in free space, \( \Gamma \) is independent of \( P \) and \( E \) is restricted to on-shell energy values,

\[
E = \frac{p^2}{2m} + (p_3 + Q)^2/2m.
\]

As in (16) for \( \chi_0 \), we rearrange the full \( \chi \) in (22) (see the Appendix) so that \( p_1 \) and \( p_3 \) are restricted to lie within the Fermi sea \( \left[ |p_1| < p_F, |p_3| < p_F \right] \) in \( ^3 \)He or to low values in \(^4\)He. We also show that the dominant term has a total energy \( E = \epsilon_1 + \epsilon_3 + \omega \). At high \( Q \) \( (Q >> p_F \) in \( ^3\)He and \( Q^2 >> (p^2) \) in \( ^4\)He) and for \( \omega \sim \omega_F \), we then have \( \mathbf{k} \approx Q/2, \mathbf{k'} \approx -Q/2 \), and \( E \approx \omega \) so that

\[
\Gamma(1+Q,3;1,3+Q) = \Gamma(Q,\omega).
\]

In this limit \( I = I(Q,\omega) \) is independent of the legs 1 and 3. We may then do the integrations directly in (22) and (23) so that (23) reduces to the RPA expression (13). Below we develop two models for the single-particle energies \( \epsilon_p \) entering \( \chi_0 \) and for \( I(Q,\omega) \).

To determine the dependence of the interaction (26) and (27) on spin and statistics, we note that this interaction, as it appears in the integral equation (23), is the spin-symmetrized interaction,

\[
I_{\lambda_1\lambda_2}^{\text{sym}}(12,34) = I^D(12,34) \pm \delta_{\lambda_1\lambda_2} I^E(12,34).
\]

Here \( I^D \) and \( I^E \) denote the direct and exchange terms, respectively, and the upper (lower) sign refers to bosons (fermions). The corresponding \( T \) matrix in (27), in terms of \( k = (p_1 - p_2)/2 \) and \( k' = (p_3 - p_4)/2 \), is

\[
\Gamma_{\lambda_1\lambda_2}^{\text{sym}}(k,k') = \Gamma_D^\pi(k,k') \pm \delta_{\lambda_1\lambda_2} \Gamma_E^\pi(k,k'),
\]

where we have suppressed the dependence on \( P \) and \( E \). To implement the approximation (28), we actually took \( k = k' = Q/2 \) in (27) so that we use the forward-scattering amplitude. Expanding \( \Gamma \) in its angular-momentum components, we have

\[
\Gamma^\pi(k,k) = \sum_I (2I + 1) \Gamma^I(k,k) = \alpha_+ + \alpha_-
\]

FIG. 1. (a) Diagrammatic equation for the two-point function \( \chi(12,Q) \); (b) the interaction \( I \) represented as a \( p-H \) and a \( p-p \) interaction.
\[ \Gamma^E(k, -k) = \sum_j (2l + 1)(-1)^l \Gamma_j(k, k) = a_e - a_o, \]

where
\[ a_e = \sum_{\text{even}} (2l + 1) \Gamma_j(k, k), \]

and \( a_o \) is a similar sum over odd-l components.

For \(^3\)He, we need the spin-symmetric interaction (19) in \( \chi_0 \),
\[ I_s(Q, \omega) = \Gamma^s(k, E) = \frac{1}{2}(\Gamma_1^s + \Gamma_2^s) = \frac{1}{2}(3a_o + a_e), \tag{30a} \]
and the spin-antisymmetric interaction (20) in \( \chi_f \),
\[ I_a(Q, \omega) = \Gamma^a(k, E) = \frac{1}{2}(\Gamma_1^a - \Gamma_2^a) = \frac{1}{2}(a_o - a_e), \tag{30b} \]
where \( k = Q/2 \) and \( E = \omega \). For spinless \(^4\)He the symmetrized interaction is
\[ I^{4s}(Q, \omega) = \Gamma^{4s}(k, E) = \Gamma^D + \Gamma^E = 2a_e. \tag{30c} \]

In the case of \(^4\)He, the interaction is unchanged if we take \( k' = \pm Q/2 \).

IV. MODELS FOR \( \chi(Q, \omega) \)

In this section we develop two models for \( \chi_0(Q, \omega) \) and the interaction \( I(Q, \omega) \) appearing in the RPA expression (13).

A. Model 1: Free particles

In model 1 we assume that the particles are noninteracting except for the explicit interaction \( I(Q, \omega) \) in (13). The input is the density \( n = N/\Omega \) and the interatomic potential \( v(r) \). The \( n \) tells us that the fluid is dense, with an average interatomic spacing not much greater than the core radius of \( v(r) \). In \( \chi_0 \) we take \( e(1) = e(0) = p^2/2m \) and the \( n_p \) is the Bose function for free particles in \(^4\)He while \( n_p = \Theta(p_F - p) \) is the Fermi function for independent particles in \(^3\)He. In each case \( I(Q, \omega) \) is approximated by the \( T \) matrix for particles scattering in free space. Following the arguments that led to (27) and (28), this is \( I_0(Q, \omega) = \Gamma_0(k, k) \) where \( k = Q/2 \), and the energy is set at its on-shell value \( E = E_{\text{rel}} + E_{\text{c.m.}} \) where \( E_{\text{rel}} = \hbar^2 k^2/m \). The free \( T \) matrix satisfies
\[ \Gamma_0(k, k') = v(k - k') + \frac{1}{\Omega} \sum_p v(k - p) \]
\[ \times \frac{1}{(\hbar^2/m)(k^2 - p^2) + i\eta} \]
\[ \times \Gamma_0(p, p'). \tag{31} \]

This was solved by expanding \( \Gamma_0(k, k) \) in its partial-wave components, \( \Gamma_0^j(k, k) \). The interactions in (30) were obtained in terms of these free-particle partial wave components.

We can also use the free-particle scattering amplitudes \( \Gamma_0 \) to calculate the total cross section \( \sigma(Q) \) for two He atoms scattering in free space. These are given by the optical theorem and the imaginary part of \( \Gamma \) as
\[ \sigma_{4s}(Q) = -\frac{2}{Q} \Gamma^{'4s}(Q), \tag{32} \]
\[ \sigma_{3s}(Q) = -\frac{2}{Q} \Gamma^{'3s}(Q). \]

In the optical theorem and for the interactions we have used the forward-scattering amplitude \( (k', k') = 1 \)
\[ \Gamma_0^d(k, k') = \Gamma_0 \left[ \frac{Q}{2} \cdot \frac{Q}{2} \right] \equiv \Gamma_0(Q). \]

B. Model 2: Galitskii-Feynman-Hartree-Fock approximation

As a second model for \(^3\)He, we use the Galitskii-Feynman-Hartree-Fock (GFHF) approximation. In this model the single-particle energies \( \epsilon_j(1) = \epsilon_j(p, \omega_1) \) appearing in \( \chi_0 \) are evaluated in the first-order or Hartree-Fock (HF) approximation. The interaction appearing in this first-order energy, \( \epsilon_j(1) \), is the Galitskii-Feynman \( (GF) \) \( T \) matrix. We also use the GF \( T \) matrix as the interaction \( I(Q, \omega) \) appearing in (13).

Explicitly, in the GFHF model the dynamics of the particles is described by the usual single-particle Green function,
\[ G_{\lambda_1}(p, \omega_1) = G_1(1) = 1/(\omega_1 - \epsilon_1(1)). \]

Here
\[ \epsilon_1(1) = \frac{p^2}{2m} + \Sigma_1(1), \tag{33} \]
and the self-energy \( \Sigma_1(1) \) is limited to the Hartree-Fock or first-order result,
\[ \Sigma^\text{HF}(1) = -i \int d^2 G_2(2) \Gamma_{1212}(12, 12). \tag{34} \]

The \( \Gamma^G \) is the symmetrized interaction
\[ \Gamma_{1234}^G(12, 34) = \Gamma^G(12, 34) \delta_{\lambda_1 \lambda_2} \delta_{\lambda_3 \lambda_4} \]
\[ - \Gamma^E(12, 34) \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4}, \tag{35} \]
which is a difference between the direct ($\Gamma^D$) and the exchange ($\Gamma^E$) terms. The GFHF is completed by using the GF $T$ matrix for $\Gamma$. The direct term of $\Gamma$ is, for example,

$$\Gamma^D(12,34) = v(p_1-p_3) + i \int d5 v(p_1-p_3) G_{\lambda_3}(5) \times G_{\lambda_6}(6) \Gamma^D(56,34).$$ (36)

Here $v(q)$ is the Fourier transform of the interatomic potential, for which we used the HFDHE2 potential of Aziz et al.\textsuperscript{37} The

$$\Gamma^D(12,34) = v(p_1-p_3) + \frac{1}{\Omega} \sum_{p_5} v(p_1-p_5) \left\{ \frac{(1-n_5)(1-n_6)}{E-\epsilon_5-\epsilon_6+i\eta} - \frac{n_5n_6}{E-\epsilon_5-\epsilon_6-i\eta} \right\} \Gamma^D(56,34).$$ (37)

The term in $n_5n_6$ represents scattering to intermediate particle states ($p_5$ and $p_6 > k_f$) and the term in $n_5-n_6$ represents scattering to the hole states ($p_5$ and $p_6 < k_f$). The $\epsilon_5$ and $\epsilon_6$ are the GFHF energies (33) so that Eqs. (33), (34), and (37) must be solved iteratively until consistent. Details of the solution are discussed in Ref. 36.

In Fig. 3 we show the on-shell GFHF $\epsilon(p_e,p_e) = \epsilon'_p + i\epsilon''_p$ up to $p \sim 10$ Å$^{-1}$. The real part, $\epsilon'_p$, can be represented approximately, by $\epsilon'_p = p^2/(2m^* + U)$, where $m^*$ is an effective mass slightly less than the bare mass $m (m^*/m \approx 0.95)$ and $U$ is a constant. In $\chi_0$ the $U$ will approximately cancel between $\epsilon_p + Q$ and $\epsilon_p$ so that the chief effect of the interaction on the real part is to introduce an effective mass $m^* \approx 0.95m$. The imaginary part $\epsilon''_p$ oscillates with $p$ at high $p$. These oscillations come from oscillations in $\Gamma''$. Indeed, from (33) and (34) we find the imaginary part of the on-shell $\epsilon'_p + Q$, for $p_1 + Q > 2p_F$, is\textsuperscript{36}

$$\epsilon''_{p+Q} = \int \frac{dp_2}{(2\pi)^3} \Gamma''(1+Q,2;1+Q,2)\Theta(p_F-p_2),$$ (38)

where $\Gamma''$ is the imaginary part of (37). The integral over $p_2$ in (38) is confined to a sphere of diameter 2$\pi p_F \sim 1.6$ Å$^{-1}$. As previously noted in (27), $\Gamma(12,34) = \Gamma(k,k';P)$ where $k = (1-2)/2$, $k' = (3-4)/2$, and $P = 1+2$. At high $Q$, the interaction in (38) is therefore

$$\Gamma''(1+Q,2;1+Q,2) \approx \Gamma'' \left[ \frac{2}{2}, \frac{Q}{2}, \frac{Q}{2} \right]$$

with $k = k' = Q/2$ and (38) becomes

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Single-particle energy $\epsilon_p$ in $^3$He; $\epsilon'_p$ (-----), $\epsilon''_p$ (----), and free-particle energy $\epsilon_p'$ (-----). Lower part shows the oscillations in $\epsilon_p''$.}
\end{figure}

FIG. 2. Diagrams for (a) the self-energy $\Sigma^{HF}$ and (b) the $T$ matrix $\Gamma$.
Thus we expect oscillations in $\Gamma''$ to appear in $\epsilon''$. More precisely, since $\Gamma''$ will be averaged over a sphere of diameter $2p_F$ in (38), the oscillations in $\epsilon''$ will be somewhat less pronounced than those in $\Gamma''$.

Finally, we use the GF $T$ matrix for the interaction $I$ in (22). We again make the approximations to $\Gamma$ discussed in (27) and (28) above to remove the dependence on the legs 1 and 2. Throughout we have also taken the c.m. momentum $P=0$ but have retained the full energy dependence $E=\omega$ as indicated in (28).

V. RESULTS

A. $\chi_0(Q,\omega)$

In Fig. 4 we compare the Lindhard function $\chi_0(Q,\omega)$ in $^4$He evaluated using free-particle energies $[\epsilon_0(p)=p^2/2m]$ with that obtained using the GFHF single-particle energies given by (33). The two $\chi_0(Q,\omega)$'s are denoted $\chi_{0}^{\text{HF}}(Q,\omega)$ (model 1) and $\chi_{0}^{\text{HF}}(Q,\omega)$ (model 2), respectively. In $\chi_{0}^{\text{HF}}(Q,\omega)$, on-shell values of the GFHF single-particle energies were used, i.e.,

$$\epsilon''(1+Q')\approx\Gamma''\left[\frac{Q}{2}\right]n.$$  (39)

![FIG. 4. Real and imaginary parts of $\chi_0(Q,\omega)$ divided by the density of states $(dn/d\epsilon)$ in $^4$He at $Q=5$ Å$^{-1}$. $\chi_0^{\text{HF}}(Q,\omega)$ (-- --); $\chi_0(Q,\omega)$ (— —).](image)

In both cases, the noninteracting Fermi function $n_p=\Theta(p_F-p)$ was used. From Fig. 4 we see that the imaginary part of $\chi_0^{\text{HF}}$ is somewhat smaller at $\omega-\omega_R$ and has tails reaching to large $(\omega-\omega_R)$ values. In contrast, $\chi_0(Q,\omega)$ is cut off sharply at the edges of the single-particle–hole excitation band. While $\chi_0$ is peaked precisely at $\omega_R$, $\chi_0^{HF}$ is peaked somewhat above $\omega_R$. The $\chi_0^{HF}$ is peaked above $\omega_R$ because the effective mass $m^*$ in GFHF single-particle energy $\epsilon_p=p^2/2m^*$ + $U$ is slightly less than $m$. Since $\epsilon_p$ has nearly a free-particle form (plus a constant), as shown in Fig. 3,

$$\epsilon_p^{\text{HF}}(Q,\omega)=\epsilon_p^0+Q,\epsilon_p^0.$$

The difference between $\chi_0^{HF}$ and $\chi_0$ arises chiefly from the imaginary part of $\epsilon_p$ in $\chi_0^{HF}$. As discussed in Sec. IV, $\chi_0$ is used in model 1, and $\chi_0^{HF}$ in model 2, for $S(Q,\omega)$.

![FIG. 5. Real and imaginary parts of $\chi_0^{HF}(Q,\omega)$ in $^4$He at $Q=6$ Å$^{-1}$. $\chi_0^{HF}(Q,\omega)$ (— —); $\chi_0^{HF}(Q,\omega)$ (— —).](image)

![FIG. 6. Real and imaginary parts of the free-particle $T$ matrix $\Gamma_0(Q)$ in $^4$He and $^3$He. $\Gamma_0^4(Q)$; (——); $\Gamma_0^3(Q)$ (— —). Two upper curves denote the real parts of the interactions and two lower curves the imaginary parts.](image)
In $^4$He, just as for $^3$He, we used the RPA expression of $\chi(Q,\omega)$. It is beyond the scope of the present work to perform a first-principles calculation of the self-energy and the momentum distribution $n_p$ of the $^4$He atoms in the presence of a Bose condensate. For this reason we chose to use free-particle propagators and, hence, a free-particle momentum distribution for the calculation of $\chi_0(Q,\omega)$ and the on-shell $\Gamma(Q)$. These approximations correspond to

$$\chi_0(Q,\omega) \approx \chi_0^0(Q,\omega),$$

$$\Gamma(Q) \approx \Gamma_0(Q).$$

Such a free-particle spectrum as $\epsilon_p^0$ is more likely to be representative of liquid $^4$He above $T_\lambda$ than below $T_\lambda$ where the poles in the single-particle Green function are known to have the phonon-roton dispersion relation. We therefore performed our calculation of $\chi_0^0(Q,\omega)$ at $T=3.2$ K which is just above the Bose condensation temperature for a gas of $^4$He atoms at liquid-$^4$He density. Choosing $T>T_\lambda$ has the added advantage that all condensate effects such as vertex functions are avoided.

In Fig. 5 we show $\chi_0^0(Q,\omega)$ for $^4$He atoms at $T=3.2$ K and $Q=6$ Å$^{-1}$. The imaginary part of $\chi_0^0(Q,\omega)$ has a very sharp peak because $n_p^0=n(\epsilon_p)$ is sharply peaked near $p=0$.

### B. Interaction

The scattering amplitude for two helium atoms interacting in free space, given by (30) and (31), is shown in Fig. 6. The $\Gamma_0(Q)$ depends only upon the relative wave vector $k=Q/2$ of the interacting pair. Shown is the spin-symmetric scattering amplitude $\Gamma_0$ in the case of $^3$He. We see that all the $\Gamma_0(Q)$'s oscillate with $Q$. The amplitude of oscillations is larger for $^3$He than for $^4$He, particularly, the amplitude of the oscillations in $\Gamma''$, relative to its magnitude, is significantly smaller in $^3$He than in $^4$He. From oscillations, $\Gamma''$ is approximately proportional to $Q$. This is necessary for the cross section

$$\sigma(Q)=-(2/Q)\Gamma''(Q),$$

and width $W/Q$ to be broadly independent of $Q$.

In Fig. 7 we compare the spin-symmetric ($\Gamma_0^s$) and spin-antisymmetric ($\Gamma_0^a$) scattering amplitudes for $^3$He. There we see that both the real and imaginary parts of $\Gamma_0^a$ are small. Thus, we expect the spin-dependent susceptibility $\chi(Q,\omega)$ to be similar to $\chi_0(Q,\omega)$, the numerator in (13).

The energy dependence of the GF $T$ matrix is displayed in Fig. 8. At all the $Q$ values investigated, we found that $\Gamma(Q,\omega)$ is approximately proportional to $\omega$ over the range of $\omega$ values of interest. The on-shell value is given by $\Gamma(Q,\omega_R)$. Thus, with the energy dependence

![FIG. 7. Real and imaginary parts of the $^3$He interaction. Upper Part: spin-antisymmetric part $\Gamma_0^a$; lower part: spin-symmetric part $\Gamma_0^s$.](image)

![FIG. 8. Off-energy-shell $T$ matrix at $Q=5$ Å$^{-1}$. $\Gamma(Q,\omega)$ (---), GF $T$ matrix including the Fermi-liquid effects; $\Gamma_0(Q,\omega)$ (---), free-particle $T$ matrix.](image)

![FIG. 9. Dynamic form factor $S(Q,\omega)$ calculated in model 2 in $^3$He at $Q=5$ Å$^{-1}$: $S_{00}^{3H}(Q,\omega)$ (---); $S_{ij}(Q,\omega)$ coherent dynamic form factor (---); $S_{ij}(Q,\omega)$ incoherent dynamic form factor (---); total dynamic form factor (---).](image)
included, $|\Gamma'(Q,\omega)|$ is larger at high $\omega$ than the on-shell value and smaller at lower $\omega$ than the on-shell value. In Fig. 8, we also show $\Gamma_0(Q,\omega)$ which is calculated from (31). This is the free-space scattering amplitude, excluding all Fermi-liquid effects, but with $\Gamma_0$ generalized to be energy dependent. We see that at $Q = 5 \text{ Å}^{-1}$ the full GF $T$ matrix is well approximated by $\Gamma_0$ if the energy dependence of $\Gamma_0$ is retained. This means that at $Q \gtrsim 5 \text{ Å}^{-1}$ $\Gamma$ is not very sensitive to whether (a) GFHF or free-particle energies $E_p^0$ are used, (b) whether scattering via intermediate hole states is included, and (c) scattering to states within the Fermi sea is excluded or not.

C. $S(Q,\omega)$

In Fig. 9 we display the dynamic form factors in $^3\text{He}$ contributing to (1) calculated using model 2. The $S_j(Q,\omega)$ describes noninteracting $^3\text{He}$ calculated using $\chi^0_{\text{HF}}$. The $S_c(Q,\omega)$ is the coherent dynamic form factor (2) evaluated using $\chi^0_{\text{HF}}$ and the GF $T$ matrix $I_c(Q,\omega)=\Gamma_c(Q,\omega)$ in the RPA, (13). Since $\Gamma_c$ is large and negative, $S_c$ is displaced to lower $\omega$ than $S_0(Q,\omega)$. The spin-dependent form factor $S_j(Q,\omega)$ differs little from $S_0(Q,\omega)$. This is because the spin-antisymmetric interaction $I_j(Q,\omega)=\Gamma_j(Q,\omega)$ in the RPA is small, as displayed in Fig. 7. The total $S(Q,\omega)$ given by (1) is dominated by $S_j(Q,\omega)$.

The $S(Q,\omega)$ in $^3\text{He}$ for $Q=5.5$, 10.0, and 13.85 Å$^{-1}$ is shown in Fig. 10. At $Q = 5.5 \text{ Å}^{-1}$, we have compared our calculated $S(Q,\omega)$ in $^3\text{He}$ with the scattering intensity observed by Mook. The basic shape is clearly similar. At $Q = 13.85 \text{ Å}^{-1}$ we compare our $S(Q,\omega)$ with scattering intensity observed by Sokol et al. The calculated $S(Q,\omega)$ clearly has high-frequency tails. These tails are characteristic of $S(Q,\omega)$ for systems of particles interacting via a potential that has a steeply repulsive core. Tails in $S(Q,\omega)$ are found in solid and liquid $^3\text{He}$ and $^4\text{He}$ in calculations using a wide variety of methods. The $S(Q,\omega)$ obtained using models 1 and 2 are compared in Fig. 11.

In $^4\text{He}$ the $S(Q,\omega)$ obtained using model 1 is displayed in Fig. 12. At larger $Q$ ($Q \gtrsim 10 \text{ Å}^{-1}$) we see that $S(Q,\omega)$ is rather narrow and sharply peaked at $\omega_k$. This peak arises from the Bose function which peaks sharply near $p \approx 0$ at low temperature, even above the Bose condensation temperature $T_c$. Near $T_c$ the chemical potential is very small and

FIG. 10. (a) $S(Q,\omega)$ (---) at $Q = 5.5 \text{ Å}^{-1}$, scattered intensity observed by Mook (Ref. 18) at constant angle $\phi = 81.6^\circ$ (○○○). (b) $S(Q,\omega)$ at $Q = 10 \text{ Å}^{-1}$ (---); Gaussian $S(Q,\omega)$ that has a width corresponding to $\langle E_k \rangle = 13 \text{ K}$ (---); (c) $S(Q,\omega)$ at $Q = 13.85 \text{ Å}^{-1}$ (---), scattered intensity measured by Sokol et al. at constant angle $\phi = 91.8^\circ$ (○○○).

FIG. 11. Dynamic form factor $S(Q,\omega)$ in $^3\text{He}$ at $Q = 6 \text{ Å}^{-1}$. $S(Q,\omega)$ in model 1 (· · ·); $S(Q,\omega)$ in model 2 (---); $S^0_{\text{HF}}(Q,\omega)$ (−−−).
DYNAMICS OF QUANTUM LIQUIDS AT HIGH MOMENTUM

$$n_p \approx \left[ \exp(\beta p^2/2m) - 1 \right]^{-1}.$$  

At low $p$, $n_p \approx (\beta p^2/2m)^{-1}$, which is very large near $p=0$. This leads to a large narrow peak in $\chi_0$ and in $S(Q,\omega)$. The shape of $S(Q,\omega)$ shown in Fig. 12 can be analyzed in more detail. As mentioned above, the sharp peaking of $S(Q,\omega)$ is characteristic of cold, Bose systems (near $T_c$). It depends only on the fact that $n_p^B$ is sharply peaked near $p=0$ and is not a consequence of interactions between the particles (e.g., final-state interactions). The calculated $S(Q,\omega)$ is compared with the scattering intensity observed in $^4\text{He}$ in Fig. 13. The tails found at $\omega$ away from $\omega_R$ do depend on the interactions. For our frequency-independent interaction $\Gamma_0(Q)$ the low-frequency tail is more noticeable than the high-frequency tail. If we were to use a frequency-dependent interaction $\Gamma'(Q,\omega)$ which had the same general frequency behavior as the $^3\text{He}$ interaction, then the peak would be shifted slightly towards the higher frequencies and the low-frequency tail is reduced.

D. Widths and peaks

The full width at half maximum, $W(Q)$, of $S(Q,\omega)$ in liquid $^4\text{He}$ is shown in Fig. 14. The points and crosses with error bars are the observed values quoted by Martel et al. The lines marked $B$ and $B'$ are model calculations of $W(Q)/Q$ by Martel et al. These are based on a Lorentzian ($B$) and a Gaussian ($B'$) model function for $S(Q,\omega)$ each having a damping factor $\gamma(Q)$ proportional to $\sigma(Q)$ [$\gamma(Q) = n(\pi Q/2m)\sigma(Q)$]. The absolute value of the lines $B$ and $B'$ is set by adjusting $\langle p^2 \rangle$ in the model to reproduce to mean observed $W(Q)$. The unmarked solid line is the present $W(Q)/Q$ calculated using model 1.

The present $W(Q)/Q$ from model 1 clearly shows oscillations with $Q$. In general, the magnitude, period, and phase of these oscillations agree well with the observed values. Since the width of $S(Q,\omega)$ does not oscillate with $Q$, the oscillations in $W(Q)/Q$ come from oscillations in $I(Q,\omega)$ appearing in the RPA (see Fig. 6). In the present model, the oscillations in $W(Q)/Q$ arise from the interaction in the RPA rather than in a particle lifetime given by $\gamma(Q)$ in the model of Martel et al.

The $W(Q)/Q$ in model 1 decreases with $Q$. This is because as $Q$ increases, $S(Q,\omega)$ peaks at increasingly higher frequency. Our approximation of a frequency-independent interaction, which is poor at low $\omega$ and

FIG. 12. Dynamic form factor $S(Q,\omega)$ calculated in model 1 in $^4\text{He}$ at $Q=6$ and 12 Å$^{-1}$.

FIG. 13. Upper part: scattered intensity observed by Cowley and Woods (Ref. 8) at $Q=7.5$ Å$^{-1}$ (○●); $S(Q,\omega)$ in $^4\text{He}$ with resolution broadening folded in (——). Lower part: scattered intensity observed by Stirling et al. (Ref. 38) at $Q=12$ Å$^{-1}$ (+ + +); $S(Q,\omega)$ in $^4\text{He}$ with resolution broadening folded in (——).
artificially increases $W(Q)$ at low $Q$, becomes increasingly valid at high $Q$. This is discussed further when we discuss the $f$-sum rule in Sec. VI. In the impulse approximation, which is valid as $Q \to \infty$, $W(Q)/Q$ is a constant proportional to $\langle \rho \rangle$.

The lower part of Fig. 14 shows the peak position, $E(Q)$, of $S(Q,\omega)$ observed by Cowley and Woods$^8$ compared to that found here in model 1. The present model 1 clearly predicts oscillations in the peak position having period and amplitude in good agreement with the observed oscillations. The predicted $E(Q)$, however, lies below the observed value. This suggests that the interaction $\Gamma_0(Q)$ in model 1 is too strong (and negative). This strong interaction displaces the peak to lower energy below $E_R$ so that $|E(Q) - E_R|$ is larger than the observed value. In Fig. 13, for example, we see clearly that the $S(Q,\omega)$ calculated using model 1 peaks at lower $\omega$ than the observed intensity. From Fig. 14, we thus see that model 1 readily predicts oscillations in both $W(Q)/Q$ and $E(Q)$ in broad agreement with experiment. Below $Q \approx 4$ Å$^{-1}$, model 1 is not valid due to its frequency-independent interaction. The good agreement of the absolute value of $W(Q)/Q$ with experiments may be fortuitous given the difference between the absolute values of the calculated and observed $E - E_R$.

The $W(Q)/Q$ of $S(Q,\omega)$ in liquid $^3$He is displayed in Fig. 15. The circles (with error bar indicated on one point) are the widths observed by Mook.$^{15}$ The solid bar is the width observed by Sokol et al.$^{17}$ in the range $12 \leq Q \leq 15$ Å$^{-1}$ and extrapolated here to $Q = 10$ Å$^{-1}$. The dashed line is the $W(Q)/Q$ obtained from $S_0^{HF}$ calculated using $\epsilon_0^{HF}$. Although the imaginary part of $\epsilon_0^{HF}$ oscillates with $p + Q$, as depicted in Fig. 3, these oscillations are not translated into oscillations in the width of $S_0^{HF}$. The $W(Q)/Q$ obtained from $S(Q,\omega)$ calculated from models 1 and 2 are shown as solid lines in Fig. 15. These have only very slight oscillations with $Q$ which originate from the oscillations in the scattering amplitudes $I_0(Q)$ and $I(Q,\omega)$. For practical purposes, we may say that models 1 and 2 predict no observable oscil-

![FIG. 14. Width and peak position of $S(Q,\omega)$ in liquid $^4$He. Upper part: $W(Q)/Q$; data of Martel et al. (Ref. 9) (○ and ×); $B$ and $B'$ model calculations from Ref. 9; (---) present width using model 1. Lower part: $E(Q) - E_R$; data of Cowley and Woods (Ref. 8) and present model 1 (---).](image1)

![FIG. 15. Widths of $S(Q,\omega)$ in liquid $^3$He: Mook's data (○●●) and guide to eye (····); present calculations using models 1 and 2 (---); $S_0^{HF}(Q,\omega)$ (——); Sokol et al.'s data for $12 \leq Q \leq 15$ Å$^{-1}$ (—).](image2)

![FIG. 16. Comparison of $W(Q)/Q$ in model 1 for liquid $^3$He and $^4$He.](image3)
DYNAMICS OF QUANTUM LIQUIDS AT HIGH MOMENTUM

lations in \(W(Q)/Q\). The models do predict a width of magnitude in excellent agreement with that observed by Sokol et al.\(^{17}\) and in reasonable agreement with the average value \(W(Q)/Q = 2.18\) meVÅ reported by Mook.\(^{16}\) However, the models do not predict a \(W(Q)/Q\) increasing to 3 meVÅ in the region \(Q = 7\) Å\(^{-1}\).

The \(W(Q)/Q\) determined from models 1 and 2 is approximately constant in the range \(3 \leq Q \leq 10\) Å\(^{-1}\). The \(W(Q)/Q\) in \(^3\)He and \(^4\)He is compared in Fig. 16. Without the denominator in (13) the \(W(Q)/Q\) for \(x_0^H\) and \(x_0^{HF}\) are 1.56 and approximately 1.8 meVÅ, respectively.

As seen from Figs. 9 and 11, the interaction displaces the peak of \(S(Q,\omega)\) to lower \(\omega\). We found no significant oscillation or variation of this displacement with \(Q\) in liquid \(^3\)He. The displacements shown in Figs. 9 and 11 are also much smaller (a factor of 3–5) than those observed by Mook.\(^{18}\)

**VI. DISCUSSION**

A central aim here is to explore how well \(S(Q,\omega)\) can be described from first principles using a \(T\)-matrix interaction in the range \(3 \leq Q \leq 15\) Å\(^{-1}\). This momentum-transfer range may be called *intermediate*. For \(Q \geq 3\) Å\(^{-1}\) liquid \(^3\)He and \(^4\)He do not support collective excitations. Thus the neutron excites single-particle (or single \(p-h\)) excitations. These excited particles interact with their immediate neighbors in the fluid. This interaction provides the interaction \(I(1+Q,2;1,2+Q)\) in (22) for \(x(12,Q)\) and \(I(Q,\omega)\) in the RPA (13). For \(Q \leq 15\) Å\(^{-1}\), we expect these interactions to be important.

We do not expect, for example, that the impulse approximation to \(S(Q,\omega)\) to be even approximately valid. Our aim is specifically to explore the effects of the interaction and not to test the validity of the IA\(^{3}\).

The chief assumption here is that the interparticle interaction can be approximated by scattering between pairs of atoms via \(v(r)\). Interactions induced via collective excitations and many-body interactions are ignored. The pair interaction is calculated in the \(T\)-matrix approximation from \(v(r)\). We have also made approximations which are valid for \(\omega \sim \omega_R\) but which may not be accurate at large \(\omega - \omega_R\). For example, the energy dependence of \(T\) matrix was represented at all \(\omega\) by the term which dominates at \(\omega = \omega_R\). Also, in \(x_0^{HF}\), we used single-particle energies at their *on-shell* energy values. In principle, the full energy dependence of these energies, as set out in (14) and (15), should be retained.

In \(^3\)He, the general shape and width of \(S(Q,\omega)\) agrees quite well with the scattering intensity profiles observed by Sokol et al.\(^{16}\) and by Mook shown in Figs. 10(a) and 10(c). The magnitude of the calculated width \(W/Q\), as shown in Fig. 15, in models 1 and 2 agrees well with the average observed value. The width predicted by \(x_0^{HF}(Q,\omega)\) alone and using \(n_p=\Theta(p_F-p)\) and \(e_p^2=p^2/2m\) is \(W/Q = \sqrt{2}n_F = 1.56\) meVÅ. The interaction \(I_0(Q,\omega)\) = \(I_0(Q,\omega)\) in model 1 increases this width to \(W/Q \approx 2.0\) meVÅ (see Fig. 15). The width of \(x_0^{HF}(Q,\omega)\), which uses \(n_p=\Theta(p_F-p)\) and dressed single-particle energies (33), is approximately 2.0 meVÅ. The \(I(Q,\omega)\) = \(I_0(Q,\omega)\) in model 2 reduces this width slightly to \(W/Q = 1.8\) meVÅ. Thus the width is set chiefly by the Doppler or kinetic broadening. The interaction in the RPA plays some role, an observable role, but a minor one. There is also cancellation of contributions from the \(e_p^2\) and \(I(Q,\omega)\). Broadly, the RPA (13), with GFHF energies (33) and the \(T\)-matrix interaction (36), describes \(S(Q,\omega)\) quite well for \(3 \leq Q \leq 15\) Å\(^{-1}\).

We found no significant oscillations in the peak position, \(E(Q)\), of \(S(Q,\omega)\) with \(Q\) in \(^3\)He. This was investigated in some detail and only weak oscillations were obtained, of the same magnitude of the oscillations of \(W(Q)/Q\) shown in Fig. 15. These weak oscillations originate from oscillations in \(I(Q,\omega)\) in the RPA. Their amplitude was 3–5 times smaller than the variation of \(E(Q)\) with \(Q\) reported by Mook.\(^{18}\)

In \(^4\)He, the present model reproduces the observed oscillations in the peak position, \(E(Q)\), and in the width, \(W(Q)/Q\), of \(S(Q,\omega)\). The magnitude of \(W(Q)\), at least at lower \(Q\), also agrees well with experiment. As noted above the absolute value of \(W(Q)/Q\) falls below the observed value as \(Q\) increases because we have used a free-boson momentum distribution \(n_p\), which is too narrow.

Thus as the interaction becomes relatively less important as \(Q\) increases, \(W(Q)/Q\) fails in the present model. The overall shape of \(S(Q,\omega)\) is also in reasonable agreement with experiment especially at high \(Q\). Particularly, a narrow profile is predicted at high \(Q\). This is due to the sharp peaking of the Bose function at low momentum values in a cold Bose gas. It is not due to the interaction. This peaking is likely to be broadened somewhat by interaction in a more complete model. Model 1 fails for \(Q \leq 4\) Å\(^{-1}\) because \(\Gamma_0\) is independent of \(\omega\). The true \(\Gamma_0\) should decrease in magnitude with \(\omega\) (as seen in Fig. 8 for \(^4\)He). Using a constant \(\Gamma_0\) leads to unrealistically large values of \(S(Q,\omega)\) at low \(\omega\), at low \(Q\). However, model 1 with a \(T\)-matrix interaction does describe \(^4\)He remarkably well at \(4 \leq Q \leq 12\) Å\(^{-1}\).

The present model 2 does predict tails in the wings of \(S(Q,\omega)\) larger than given by a Gaussian function [see Fig. 10(b)]. In model 2, these originate from tails in \(x_0^{HF}\) and from the interaction \(\Gamma(Q,\omega)\). Model 1 predicts large tails at low \(\omega\) as shown in Figs. 11 and 12(a).

In model 1, these arise from using a frequency-independent interaction \(\Gamma_0(Q)\), which is complex and too large at small \(\omega\). Thus the large value of \(S(Q,\omega)\) in model 1 near \(\omega = 0\) is an unrealistic artifact of using \(\Gamma_0(Q)\) in \(\Gamma(Q,\omega)\) for \(\omega \rightarrow 0\) and should be discounted. The tails in model 2 are more realistic but not entirely reliable because the full frequency dependence of \(e_p^2\) in \(x_0^{HF}\) has not been retained and the frequency dependence of \(\Gamma_0(Q)\) was selected to be most accurate near \(\omega = \omega_R\) as discussed in the Appendix. However, there is substantial evidence\(^{23}\) that \(S(Q,\omega)\) does indeed have high-frequency tails in liquid and solid helium.

It is clear from the RPA expression (13) that if the interactions \(I_{a,b}(Q,\omega)\) satisfy the limit

\[
\lim_{\omega \rightarrow \infty} \frac{I_{a,b}(Q,\omega)}{\omega^2} = 0,
\]
the denominator of the RPA expression goes to 1 and the susceptibilities \( \chi_{c,t} \) satisfy the high-frequency limit
\[
\chi_{c,t}(Q,\omega) \rightarrow \chi_0(Q,\omega) \quad (\omega \rightarrow \infty).
\]

In model 1, where \( \chi_0 \) is simply the noninteracting susceptibility, \( \chi_0 \) satisfies the \( f \)-sum rule and consequently \( \chi_{c,t} \) also satisfies the high-frequency form of the \( f \)-sum rule:
\[
\chi_{c,t}(Q,\omega) \rightarrow \frac{2n\omega R}{\omega^2} \quad (\omega \rightarrow \infty). 
\]  
(40)

In model 2, \( \chi_0 = \chi_0^{HF} \) only satisfies the \( f \)-sum rule for atoms of effective mass \( m^* \approx 0.95m \) and our calculation of \( \chi \) deviates from the \( f \)-sum rule accordingly.

The \( f \)-sum rule can also be written as the first moment integral of \( \chi'' \):
\[
-\int_0^\infty \frac{d\omega}{\pi} \chi''(Q,\omega) = n\omega R .
\]  
(41)

This second form of the \( f \)-sum rule is equivalent to (40) as \( \chi \) generally satisfies the Kramers-Kronig (KK) relation:
\[
\chi(Q,\omega) = \int_0^\infty \frac{d\omega'}{\pi} \frac{\chi''(Q,\omega')}{\omega' - \omega - i\eta} ,
\]  
(42)

which relates the real part of \( \chi \) to its imaginary part.

If \( \chi \) is to be calculated from an RPA expression such as (13), the interaction \( \Gamma(Q,\omega) \) must satisfy some rather restrictive requirements in order for the \( \chi(Q,\omega) \) to satisfy the KK relation (42). First of all \( \Gamma(Q,\omega) \) must have the same symmetry as \( \chi(Q,\omega) \):
\[
\Gamma(Q,\omega) = \Gamma^*(Q,-\omega) .
\]  
(43)

Secondly, for all positive \( \omega \), \( \chi'' \leq 0 \) and hence from (13) we need
\[
\chi''(Q,\omega) \leq -\frac{\chi_0''(Q,\omega)}{|\chi_0(Q,\omega)|^2} ,
\]  
(44)

for all \( \omega > 0 \). Finally, the full \( \chi \) must be a retarded function and hence the poles of
\[
[1 - \Gamma(Q,\omega)\chi_0(Q,\omega)]^{-1}
\]
must be below the real \( \omega \) axis. In any numerical calculation, this last condition is the most difficult to verify.

The symmetry of \( \Gamma(Q,\omega) \), (43), requires that the \( \Gamma''(Q,\omega) \) be odd in frequency and \( \Gamma''(Q,\omega) \) should vanish as \( \omega \rightarrow 0 \). It is clear that our frequency-independent interaction \( \Gamma_0(Q) \) used in model 1 for both \( ^3\text{He} \) and \( ^4\text{He} \) does not satisfy this requirement. Hence our RPA \( \chi \) will not satisfy the KK relation at low frequencies and it follows that \( \chi \) will not satisfy the integral form (41) of the \( f \)-sum rule. This deviation from the \( f \)-sum rule is most apparent at low \( Q \) where \( \chi_0 \) is peaked at relatively low \( \omega \), where the symmetry requirement for the \( \Gamma''(Q,\omega) \) is most important. At the higher-\( Q \) values the \( \chi \) calculated from model 1 satisfies the integral \( f \)-sum rule to better than 10% in \( ^4\text{He} \) if we integrate up to \( 2\omega R \) and to better than 5% in \( ^3\text{He} \).

In model 2, the frequency dependence of \( \Gamma(Q,\omega) \) implies that both (43) and (44) are satisfied. While this does not necessarily mean that the real and imaginary parts of the RPA \( \chi \) are related by the KK relation, the main deviation to the integral form of the \( f \)-sum rule comes from the fact that \( \chi_0 \) is calculated with an effective mass different from the bare mass. As a consequence, model 2 satisfies the integral \( f \)-sum rule to better than 20%.

The second moment of \( S(Q,\omega) \) is also of special interest, since a reliable calculation of it would yield information on the \( \langle E_R \rangle \) of the atoms in the liquid. We have not been able to calculate the second moment of \( S(Q,\omega) \) with sufficient accuracy in either of the present models. In model 1, the imaginary part of the interaction that enters the RPA contributes to the tails of \( S(Q,\omega) \). In model 2, not only the frequency-dependent interaction but the numerator of RPA (\( \chi_0^{HF} \)) also contributes at large \( \omega \). The \( \chi_0^{HF}(Q,\omega) \) should be calculated using the off-shell energies \( \epsilon_0(\omega) \) instead of \( \epsilon_0(\omega) \) as we used here. It is noted that this method would preserve the moments of \( S(Q,\omega) \).

It is interesting that the present models reproduce the observed oscillations in \( W(Q)/Q \) in \( ^3\text{He} \), but predict very small oscillations in \( ^4\text{He} \). This, we believe, may be understood as a combination of two physical effects. Firstly, the average of Doppler \( W(Q)/Q \) is larger in \( ^3\text{He} \) than in \( ^4\text{He} \). This is \( W(Q)/Q \approx 22-25 \) K in \( ^3\text{He} \) versus \( W(Q)/Q \approx 18-22 \) K in \( ^4\text{He} \). This follows from the larger kinetic or zero-point energy in \( ^3\text{He} \) than in \( ^4\text{He} \). The larger Doppler \( W(Q)/Q \) tends to mask any variations in the contribution to \( W(Q)/Q \) coming from interactions. Secondly, the oscillations in the scattering amplitude \( \Gamma(Q,\omega) \) have a somewhat smaller amplitude in \( ^3\text{He} \) than in \( ^4\text{He} \) (cf. Fig. 6). This is particularly true of the imaginary part \( \Gamma''(Q,\omega) \) which is most important. The smaller-amplitude oscillations in \( \Gamma(Q,\omega) \) lead to smaller-amplitude glories in the \( \text{He-He} \) atom scattering cross section, which we show in Fig. 17 for \( ^3\text{He} \)-\( ^4\text{He} \) and

![Fig. 17. Total cross sections \( \sigma(Q) \) in \( ^3\text{He} \) \((+++)\) and in \( ^4\text{He} \) \((\ldots\ldots)\) (Ref. 11); calculated cross sections \( ^3\text{He} \) \((-\ldots-)\) and \( ^4\text{He} \) \((-\ldots-)\).](image-url)


4He-3He scattering. The solid and dashed lines in Fig. 17 are calculated from our \( \Gamma_0(Q) \) using the optical theorem (32). [The error between our calculated \( \sigma(Q) \) and the observed value at higher \( Q \) is believed to be due to error in \( v(r) \).] Thus the contribution to \( W(Q)/Q \) from the interaction can be expected to oscillate with a smaller amplitude in 3He. Contributions to \( W(Q)/Q \) from different sources add in a very nonlinear way, as a sum of squares for Gaussian functions, for example. Thus we believe, the combination of a larger Doppler \( W(Q) \) and smaller-amplitude oscillations in \( \Gamma''(Q,\omega) \) means the \( W(Q)/Q \) oscillates with a very much smaller amplitude in 3He.

Martel et al.\(^9\) have reproduced the oscillations in \( W(Q)/Q \) in liquid 4He using an interesting model. The \( S(Q,\omega) \) is written as

\[
S_M(Q,\omega) = \int dp n(p) \frac{1}{\pi} \frac{\gamma'(p+Q)}{(\omega - \omega_R - (Q-p)/m)^2 + \gamma^2(p+Q)},
\]

(45)

where \( \gamma \) is related to the He-He atom cross section \( \sigma(Q) \) by\(^10\)

\[
\gamma'(Q) = n \frac{\hbar Q}{2m} \sigma(Q) .
\]

(46)

They used the observed 4He-3He atom cross section \( \sigma(Q) \) (see Fig. 17) and an \( n(p) \) with \( \langle p^2 \rangle \) adjusted so that (45) reproduces the observed magnitude of \( W(Q)/Q \). This produces the line B shown in Fig. 14. In (45) the oscillations in \( W(Q)/Q \) follow from the oscillations in \( \sigma(Q) \) shown in Fig. 17 which suggests, as noted by Martel et al., that the oscillations (45) come from oscillations in the single-particle lifetimes rather than from the interaction \( \Gamma(Q,\omega) \) in the present RPA model.

The \( S_M(Q,\omega) \) is very similar to \( S_0^{\text{HF}}(Q,\omega) \) discussed for 3He. Indeed (45) and (46) may be derived from our

\[
S_0^{\text{HF}}(Q,\omega) = -(n\pi)^{-1}X_0^{\text{HF}}(Q,\omega) ,
\]

where \( X_0^{\text{HF}} \) is given by (18). To arrive at (45) from (18) we approximate the real part of \( \epsilon_{p+Q} - \epsilon_p \) of the denominator in (15) by its free-particle value and define

\[
\gamma'(p+Q) = - (\epsilon_{p+Q}' - \epsilon_p')/\hbar .
\]

To obtain (46) we assume \( \epsilon_{p}' \) is small (since \( p \approx p_F \) ) and use the high-\( Q \) values (38) and (39) for \( \epsilon_{p+Q}' \) giving

\[
\epsilon_{p+Q}' = \Gamma''(p+Q) n .
\]

Using the optical theorem (32) \( [\Gamma''(Q) = -(\hbar^2 Q/2m)\sigma(Q)] \), then leads directly to

\[
\gamma'(Q) = \epsilon_{p+Q}'/\hbar = n(\hbar Q/2m)\sigma(Q) .
\]

In this relation the frequency dependence of \( \Gamma''(Q) \) is ignored. As shown in Fig. 15, \( S_0^{\text{HF}}(Q,\omega) \) does not provide oscillations in \( W(Q)/Q \) in 3He, even though our \( \Gamma''(Q,\omega) \) reproduces \( \sigma(Q) \). We have evaluated the model of Martel et al., (45) and (46), using \( n(p) = O(p_F - p) \) and the 3He-3He \( \sigma(Q) \) shown in Fig. 17. The resulting \( W(Q)/Q \) is compared with our \( S_0^{\text{HF}}(Q,\omega) \) values in Fig. 18. There we see the oscillations predicted by \( S_0^{\text{HF}}(Q,\omega) \) and \( S_0^{\text{HF}}(Q,\omega) \) are indeed similar for \( Q \approx 5 \text{ Å}^{-1} \). Comparing the scales of Figs. 15 and 18, we see that neither the single-particle lifetime model of Martel et al. nor ours predict observable oscillations in 3He. This is again due to the broader \( n_p \) and smaller oscillations in \( \sigma(Q) \). Given these general arguments, we do not expect significant oscillations in \( W(Q)/Q \) in 3He.

Without going into a detailed calculation of the self-energy of the 4He liquid, what other model calculations are possible for liquid 4He? One might suggest that a calculation based on a realistic (phonon-roton) spectrum and/or a realistic momentum distribution would be reasonable alternatives to a free Bose gas \( \chi_0(Q,\omega) \). Such calculations are not without difficulties. First of all, one does not have absolute freedom in choosing a quasiparticle spectrum and momentum distribution. The expression for \( S(Q,\omega) \) should satisfy a detailed-balance condition and \( S(Q,\omega) \) should be non-negative for all \( \omega \). At a fundamental level, both the quasiparticle spectrum and the momentum distribution are determined by the same single-particle Green function and, hence, are not independent. Secondly, the artificial injection of a realistic spectrum and/or momentum distribution into our RPA-like calculation implies bringing into the calculation a host of new diagrams the structure of which we know nothing. This could lead to serious overcounting. For example, using a \( T \)-matrix interaction \( \Gamma(Q,\omega) \) the imaginary part of which grows linearly with \( Q \), we would be overcounting if we used a realistic momentum distribution \( n_p \) in \( \chi_0 \) because the end result for \( \chi \) at high \( Q \) would not satisfy the impulse approximation with the same momentum distribution \( n_p \). In other words one should not use the true momentum distribution for the calculation of \( \chi_0 \).

Another possible model calculation for liquid 4He starts with the Bogoliubov-model expression for \( \chi_0 \) instead of the Lindhard expression (14). Unfortunately

![FIG. 18. \( W(Q)/Q \) in liquid 4He predicted by \( S_0^{\text{HF}}(Q,\omega) \) (---) and the Martel et al. model (—).](image)
this model involves the Bose condensate and coherence factors $u_p$ and $v_p$ which are only known in the Bogoliubov limit: the weakly interacting dilute Bose gas at low temperature. We did attempt some model calculations based on this Bogoliubov model and a realistic (phonon-roton) spectrum. Unfortunately no obvious approximation to the coherence factors allows the Bogoliubov $\chi_0$ to satisfy the $f$-sum rule with a condensate fraction which is realistic for liquid $^4$He.

To improve the present calculations, the first step is probably to include the energy dependence of the single-particle energies as indicated in (33). This would improve the accuracy of $S(Q,\omega)$ at low $\omega - \omega_R$ and make calculations of the moments of $S(Q,\omega)$ more reliable. Also, improved $\epsilon_p$ at low $p$ which include contributions from induced interactions beyond the GFHF approximation would be interesting. This, however, may not change $S(Q,\omega)$ at high $Q$ very much. As noted in Fig. 5, we found that for $Q > 5$ $\AA^{-1}$ the full $\Gamma(Q,\omega)$ could be quite well approximated by the $\Gamma_0(Q,\omega)$ for two atoms scattering in free space. This tells us that Fermi-liquid effects, the presence of the Fermi sea, and use of dressed $\epsilon_p$ in the Bethe-Salpeter equation are not very important. In the present calculations we have taken the c.m. momentum $P=0$. We found that setting $P=Q$ does change $\Gamma(Q,\omega)$ somewhat at $Q=5$ $\AA^{-1}$, but the resulting $S(Q,\omega)$ does not change significantly on the scale of accuracy sought here. Thus improving the accuracy of $\Gamma(Q,\omega)$ will not affect $S(Q,\omega)$ greatly at high $Q$. It would be interesting to extend model 2 to liquid $^4$He.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge helpful discussions with Drs. A. Griffin, P. E. Sokol, H. Mook, E. C. Svensson, and W. G. Stirling. This work has been supported by the U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. DOE 284ER45082.

APPENDIX

We explore the energy dependence of the interaction $I$ in the general equation for $\chi(12;Q)$ in (22). The aim is to show that the dominant terms in $I$ have the energy dependence given by (28). This appendix follows closely the development of $\chi(12;Q)$ by Green et al.\textsuperscript{35}

From (23) and (22) the dynamic susceptibility is

$$\chi(Q)=\int d1 \int d2 X(12;Q)$$

$$=\chi_0(Q)+\int d1 \chi_0(1,1) \int d3 \Omega(1+Q,3,1,3+Q) \times \int d2 \chi(32,Q).$$

(A1)

This may be solved by iteration as

$$\chi(Q)=\chi_0(Q)+\int d1 X_0(1,1) \Omega \times \int d3 \Omega(1+Q,3,1,3+Q) \times \chi_0(3,3)+\cdots$$

$$=\chi_0(Q)+\chi_0(Q)I_2(Q)\chi_0(Q)+\cdots, \quad (A2)$$

where

$$I_2(Q)=\int d1 \int d3 \chi_0(1Q)$$

$$\times I(1+Q,3,1,3+Q) \times \chi_0(3) = (A3)$$

Approximately we then have\textsuperscript{41}

$$\chi(Q) \approx \chi_0(Q) \frac{1}{1-I_2(Q)\chi_0(Q)} .$$

(A4)

The aim here is to use the second-order expression for $I(Q)$ to explore the energy dependence. This is clearly not exact, but it will allow us to identify the dominant terms.

The $\Gamma(1+Q,3;1,3+Q)$ is approximated by a $T$ matrix,

$$\Gamma(1+Q,3;1,3+Q)$$

$$=\Gamma(p_1+Q, p_3, p_1, p_3; Q; E = \omega_1+\omega_3+\omega),$$

which depends upon the total energy $E$. The $T$ matrix has the same analytic properties as the two-body Green function. A spectral representation is

$$\Gamma(E)=\int_0^\infty \frac{dx}{2\pi} \left\{ \frac{\Gamma(x)}{E-(2\mu+x)+i\eta} - \frac{\Gamma(x)}{E-(2\mu-x)-i\eta} \right\} .$$

(A5)

The first term comes from the $p-p$ intermediate states $(2\mu+x)$ and has poles below the real axis. The second term comes from the $h-h$ intermediate states $(2\mu-x)$ and has poles above the real axis. At high $Q$, the second term is much smaller than the first (recall the hole-line expansion\textsuperscript{42}) and we neglect it. This amounts to approximating the GF $T$ matrix by the Brueckner $T$ matrix. The $\Gamma$ is then analytic above the real axis.

To evaluate the integrals over $\omega_1$ and $\omega_3$ in $\Gamma$, we replace the integrals along the real axis by a contour integral closed above the real axis. Then

$$I_2(Q,\omega)=\int \frac{d\omega_1}{(2\pi)^3} \int \frac{d\omega_3}{(2\pi)^3} I_2(Q,\omega) \times \Gamma(E = \omega_1+\omega_3+\omega),$$

(A6)

where

$$I_2(Q,\omega)=\int \frac{d\omega_1}{2\pi} \int \frac{d\omega_3}{2\pi} \chi_0(1;Q;\omega)$$

$$X_0(1;Q;\omega),$$

(A7)

and

$$X_0(1;Q;\omega)=-iG^0(1)G^0(1+Q)$$

$$=iG^0(p_1,\omega_1)G^0(p_1+Q,\omega_1+\omega),$$

(A8)
with

\[ G^0(p_1, \omega_1) = \left[ \frac{(1-n_1)}{\omega_1 - \epsilon_{1}^0 + i\eta} + \frac{n_1}{\omega_1 - \epsilon_{1}^0 - i\eta} \right]. \] (A8)

When we close the contour above the real axis, we enclose the poles in the hole terms of single-particle Green functions \( G^0 \) only, which are proportional to \( n_p \). A straightforward contour integration gives

\[
\begin{align*}
\bar{I}_2 = & \left[ n_p n_p G^0(p_1 + Q, \epsilon_1 + \omega) G^0(p_3 + Q, \epsilon_3 + \omega) \Gamma(\epsilon_1 + \epsilon_3 + \omega) + n_p n_p G^0(p_1 + Q, \epsilon_1 + \omega) G^0(p_3, \epsilon_3 + Q - \omega) \Gamma(\epsilon_1 + \epsilon_3 + Q) \\
& + n_p n_p G^0(p_1, \epsilon_1 + Q - \omega) G^0(p_3 + Q, \epsilon_3 + \omega) \Gamma(\epsilon_1 + \epsilon_3 + \omega) \\
& + n_p n_p G^0(p_1, \epsilon_1 + Q - \omega) G^0(p_3, \epsilon_3 + Q - \omega) \Gamma(\epsilon_1 + \epsilon_3 + Q - \omega) \right].
\end{align*}
\] (A9)

Here \( \epsilon_{1+Q} \) with no energy dependence indicated denotes the on-shell value

\[ \epsilon_{1+Q} = \epsilon_1 + Q(\epsilon_1 + Q). \]

We now specialize to \( Q \gg 2p_F \) for which \( n_p n_p = 0 \).

Using \( G^0 \) given by (A8) we have

\[
\begin{align*}
n_p G^0(p + Q, \epsilon_1 + \omega) = & \left[ \frac{n_p}{\epsilon_p + \omega - \epsilon_p + Q(\epsilon_1 + \omega) + i\eta} \\
= & \chi_1(p, \omega). \end{align*}
\]

Here \( \chi_1(p, \omega) \) is the first term of \( \chi_0 \) in (17) which is the dominant term of \( \chi_0 \) for \( \omega \sim \omega_R \). Also,

\[
\begin{align*}
n_p n_p G^0(p + Q, \epsilon_1 + \omega) = & \left[ \frac{n_p Q}{\epsilon_p + Q - \epsilon_p + Q(\epsilon_1 + \omega) + i\eta} \\
= & \chi_2(p, \omega). \end{align*}
\]

The \( \chi_2(p, \omega) \) is the second term of \( \chi_0 \) in (17). The \( \chi_2''(p, \omega) = 0 \) for \( \omega > 0 \) and \( \chi_2' < \chi_1' \) for \( \omega \sim \omega_R \). The full \( \chi_0' \) and its first term \( \chi_1' \) are compared in Fig. 19. There we see indeed that \( \chi_0'(p, \omega) = \chi_1'(p, \omega) \) for \( \omega \sim \omega_R \) so that \( \chi_2' \) is small. Thus we have

\[
\begin{align*}
\bar{I} = & \left[ \chi_1(p_1, \omega) \chi_1(p_3, \omega) \Gamma(\epsilon_1 + \epsilon_3 + \omega) \\
& + \chi_1(p_1, \omega) \chi_2(p_3, \omega) \Gamma(\epsilon_1 + \epsilon_3 + Q) \\
& + \chi_2(p_1, \omega) \chi_1(p_3, \omega) \Gamma(\epsilon_1 + Q + \epsilon_3) \\
& + \chi_2(p_1, \omega) \chi_2(p_3, \omega) \Gamma(\epsilon_1 + Q + \epsilon_3 + Q - \omega) \right].
\end{align*}
\] (A10)

The dominant term in \( \bar{I} \) is clearly the first term. Our basic approximation is to assume that the energy dependence in all terms is the same as that in the first term. This will certainly be a good approximation in the region of the peak of \( S(Q, \omega) \) where \( \omega \sim \omega_R \). We then have

\[
\bar{I}(Q, \omega) = \chi_0(p_1, \omega) \chi_0(p_3, \omega) \times \Gamma(1 + Q, 3; 1, 3 + Q; E = \epsilon_1 + \epsilon_3 + \omega). \] (A11)

We note also that the dominant first term of (A10), and that \( |p_1| \) and \( |p_3| \) must lie below \( p_F \). Thus from (27) and for \( Q \gg p_F \),

\[
\Gamma(1 + Q, 3; 1, 3 + Q) = \Gamma(k, k'; P, E),
\]

with

\[
\begin{align*}
k &= (p_1 + Q - p_3)/2 \approx Q/2, \\
k' &= (p_1 - p_3 - Q)/2 \approx Q/2, \\
P &= p_1 + p_3 + Q \approx Q,
\end{align*}
\]

and \( E \approx \omega \) so that

\[
\Gamma(1 + Q, 3; 1, 3 + Q) = \Gamma \left[ \frac{Q}{2}, \frac{Q}{2}; E = \omega \right] = \Gamma(Q, \omega). \] (A12)

Thus we find that at high \( Q \gg p_F \), the dominant energy dependence that goes into the \( \bar{T} \) matrix has a total energy \( E = \omega \). At high \( Q \) we also have \( p_1 + Q - p_3 \approx Q \) and the c.m. momentum \( P = p_1 + p_3 + Q \approx Q \). The c.m. energy is therefore

\[
E_{c.m.} = P^2/4m \approx k^2 Q^2/4m = \omega_R/2.
\]

The relative energy variable is thus

\[
E_{rel} = E - E_{c.m.} \approx \omega - \omega_R/2.
\]
The on-energy-shell value of the full Fermi-liquid $T$ matrix is then $E = \omega_k$ which has an
\[ E_{rel} = \frac{\omega_k}{2} = \hbar^2 Q^2 / 4m = \hbar^2 k^2 / m \]
which coincides with the on-energy-shell value of the free-particle $T$ matrix (29).

In the present calculations we chose $\mathbf{p} = 0$ in the Fermi functions $n_\mathbf{k}$ and $n_\mathbf{k}$ in (38), but not in the energy denominator. We also retained $\mathbf{p} \neq 0$ in some calculations. Although $\Gamma$ depends upon $\mathbf{p}$ somewhat, we found $S(Q, \omega)$ and the widths $W(Q) / Q$ were insensitive to the value of $\mathbf{p}$.

22R. M. Panoff (private communication).
40The Bogoliubov model of a weakly interacting dilute Bose gas has been discussed by a number of authors including A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, Methods of Quantum Field Theory in Statistical Physics, revised edition (Dover, New York, 1965), pp. 32–36.
41This approximation was suggested to us by P. Schuck.