Spin dynamics in a polarized Bose gas

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(Received 15 August 1990)

The equation of motion for the spin density of a spin-polarized Bose gas is derived for low temperatures down to $T_c$, the critical temperature of Bose-Einstein condensation. In the classical regime, our results agree with those of Lhuillier and Laloe. However, at low $T$, $\tau_1$ and $\tau_2$ become qualitatively different, especially at larger polarization $\mathcal{M}$, and are no longer proportional to $T^{1/2}$. Particularly, $\tau_1$ reaches a maximum at $T \sim 1.6T_c$, and decreases thereafter so that intrinsic spin-wave damping in a Bose gas increases with decreasing $T$ for $T < 1.6T_c$.

At low temperature, the equation of motion for the spin density $\mathbf{m}$ in nuclear polarized paramagnetic liquids (e.g., liquid $^3$He, $^4$He-$^3$He mixtures) and gases (e.g., $^3$He, $^4$He) differs from the conventional Fick's law form $\partial \mathbf{m}/\partial t = D \nabla^2 \mathbf{m}$ of classical systems. In general, the equation becomes nonlinear and anisotropic and the transverse component has an oscillatory (wave) as well as a diffusive character. This wavelike behavior results from precession of the spins in the molecular field resulting from the exchange interaction introduced by the identical particle (anti)symmetrization. This prominent quantum effect in the spin dynamics of polarized systems has been confirmed experimentally. Specifically, standard spin-echo experiments in liquid $^3$He (Ref. 3) and $^4$He-$^3$He mixtures1 display spin-wave character and transverse spin oscillation modes (spin waves) were observed in liquid $^3$He (Ref. 5), $^4$He-$^3$He mixtures,6,7 atomic hydrogen ($^1$H) gas,8,9 and $^3$He gas.10

In his derivation of the equation for the spin density in weakly polarized Fermi liquids, Leggett1 noted, without elaboration, that the transverse spin-diffusion relaxation time $\tau_1$ is the same as the longitudinal relaxation time $\tau_1$. The equivalence of $\tau_1$ and $\tau_2$ for polarization $M \rightarrow 0$ [$M \equiv (n_+ - n_-)/(n_+ + n_-)$], where $n_+ \ (n_-)$ is the density of spin-up (down) nuclei may be justified using spin-rotation invariance alone (but, see below). On the other hand, Lhuillier and Laloe2 derived the spin-dynamical equation for nondegenerate Fermi and Bose gases by solving the kinetic equation developed by them. They found that the interference effect during the collisions (the identical spin-rotation effect) leads to a nonlinear spin hydrodynamic equation which has the same form as that derived by Leggett for Fermi liquids, with a change in sign of the exchange interaction term for Bose gases. Both the Leggett1 and Lhuillier and Laloe2 equations contain a single diffusion coefficient [i.e., $D_1 = D_\perp$, where $D_\perp$ ($D_\parallel$) is the longitudinal (transverse) diffusion coefficient, defined explicitly below]. Thus, the distinction between transverse and longitudinal diffusion was not emphasized.

In 1985, Meyerovich11 reasoned that, for degenerate polarized Fermi gases, $\tau_1$ ought to be different from $\tau_2$, and that the equation for the spin density should contain two different diffusion coefficients, $D_\parallel$ and $D_\perp$. Jeon and Mullin12 calculated $\tau_1$ and $D_\perp$ for a dilute Fermi gas from the kinetic equation they presented13 which is equivalent to the equation derived earlier by Silin.14 They showed that when

$$ T/T_F \ll [(1 + M)^{2/3} - (1 - M)^{2/3}] $$

($T_F$ is the Fermi temperature for the unpolarized gas, $\tau_1$ and $D_\perp$ differ from $\tau_2$ and $D_\parallel$ and become independent of temperature. Thus, the difference between $\tau_1$ and $\tau_2$ depends on both polarization and degeneracy. As is often the case with transport coefficients, a simple intuitive argument explaining why $\tau_1$ and $\tau_2$ differ cannot be readily given. However, it is clear that polarization, the importance of the final-state factors in the collision integral at low temperature, together with the fact that the transverse spin is a coherent linear superposition of up- and down-spin states give rise to differing behavior of $\tau_1$ and $\tau_2$. As we shall see, $\tau_1$ is significantly different from $\tau_2$ for a spin-polarized Bose gas even for temperatures above the critical temperature. For bosons, the existence of particles in final states enhances the scattering rate in contrast to fermions.

In this work, we derive equations for the spin current and density of a polarized Bose gas ($^1$H) at temperatures above $T_c$, the critical temperature of Bose-Einstein condensation of the dominant spin atoms, by solving a kinetic equation which is a generalization of that obtained by Jeon and Mullin.13 Spin-polarized atomic hydrogen $^1$H can be considered as a spin-$1/2$ Bose gas in which the spin variable is the nuclear spin but the atoms obey Bose statistics. Our derivation parallels that of Ref. 12 so that fermions and bosons can be placed on an equal footing. We evaluate $\tau_1$, $\tau_2$, $D_\perp$ ($D_\parallel$) and find that $\tau_2$, $\tau_1$ are not proportional to $T^{-1/2}$ at low temperatures where degeneracy is important. For $T < 1.6T_c$, $\tau_1$ actually decreases as $T$ decreases. Although in usual experiments8,9,15 the nuclear spin polarization $M$ is negative, in following we consider the up-spin species as dominant (positive $M$).16

Using the $T$ approximation, instead of the Born approximation, we obtain a kinetic equation for the matrix
distribution function

\[ n_p(r,t) = \frac{1}{2} \left[ f_p(r,t) + \sigma_p(r,t) \right] , \]

where \( f_p \) and \( \sigma_p \) are the particle and spin-density distribution functions, respectively. The underscore denotes a 2x2 matrix. The \( \mathbf{n}_p(r,t) \) is the Wigner transform of the one-body reduced density matrix. The equation is

\[
\frac{\partial n_p(r,t)}{\partial t} + \frac{1}{2} \left( \nabla p \cdot \nabla \mathbf{u} - \nabla \mathbf{u} \cdot \mathbf{u} - \frac{1}{2} \nabla \mathbf{u} \cdot \nabla \mathbf{u} \right) + \frac{1}{2} \left[ \nabla \Sigma_p(r) \cdot \nabla \mathbf{u} + \nabla \mathbf{u} \cdot \nabla \Sigma_p(r) \right] + \frac{1}{2} \left[ \nabla \Sigma_p(r) \cdot \nabla \mathbf{u} + \nabla \mathbf{u} \cdot \nabla \Sigma_p(r) \right] = \left[ f_p(r,t) + \sigma_p(r,t) \right] \cdot \nabla (\mathbf{u} \cdot \mathbf{u}) ,
\]

and spin density

\[ m(r,t) = \int \left( \frac{d\mathbf{p}}{h^3} \right) \sigma^0_{p(r,t)} . \]

Lhuillié and Laloo used the local equilibrium distribution function (3) with the Boltzmann (nondegenerate) \( n^0_{\text{por}} \) replacing the Fermi or Bose \( n^0_{\text{por}} \).

To derive the equation for the steady-state spin current, we substitute, as usual, the local equilibrium distribution function \( n^0_{\text{por}} \) in the drift term and linearize the collision terms (the first-order Chapman-Enskog expansion). Using variational methods, we solve this linearized kinetic equation. For simplicity, we make the s-wave approximation

\[ t(k_i,k_f) = 4\pi h^2 a / m^* \]

\( (a = 0.72 \text{ Å for } H^1) \). For \( H^1 \), the s-wave approximation is valid for \( T \leq 50 \text{ mK} \). The details of derivation for fermions have been presented previously \( 12,13 \) and the same method applies for bosons. After somewhat lengthy (but straightforward) calculation, we find that the steady-state spin current in the rotating frame for both bosons and fermions is

\[ J_s(m) = -D_1 \int \frac{\partial m}{\partial r_i} \left[ \mathbf{e} - D_1 (1 + \mu^2 m^2)^{-1} \right] \times \left[ \frac{\partial \mathbf{e}}{\partial r_i} - \eta \frac{m^*}{m} \mathbf{e} \times \left[ \frac{\partial \mathbf{e}}{\partial r_i} \right] \right] , \]

where the spin-diffusion coefficients \( D_1 \) and \( D_1 \) are given by

\[ \alpha_2 = 2 \int \sum_{\sigma} \frac{\partial n_{\text{por}}}{\partial \epsilon_{\text{por}}} \int \left[ \frac{d\mathbf{p}}{h^3} \right] \sigma \frac{\partial \mathbf{e}}{\partial \epsilon_{\text{por}}} \right] n_{\text{por}}^{-1} , \]

and \( \mu = -4\pi h a \tau_s / m^* \) is the spin-rotation parameter. The spin-relaxation times \( \tau_1 \) and \( \tau_2 \) are

\[ \tau_1 = h^2 m^* j_1 m / [4 \sinh(\Delta/2) a^2 L_1] , \]

\[ \tau_2 = h^2 m^* j_2 m / [4 \sinh(\Delta/2) a^2 L_2] . \]
In the above, \( n_+ \) (\( n_- \)) is the density of particles of up (down) spin in the local reference frame of spin space, \( n = n_+ + n_- \), \( m = n_+ - n_- \), \( \Delta = \beta (\mu_+ - \mu_-) \), while

\[
L_\parallel = \left\langle \left( \hat{p}_1 - \hat{p}_3 \right) \right. n_+^0, n_-^0, \eta_\parallel^0, \eta_\parallel^0 \left. \right| \right.
\]

and

\[
L_\perp = \sum_\sigma \exp(\frac{-\sigma \Delta t}{2}) \left\langle \left( \hat{p}_1 - \hat{p}_3 \right) \right. n_+^0, n_-^0, \eta_\perp^0, \eta_\perp^0 \left. \right| \right.
\]

are averaged collision integrals. Here, \( \eta_\parallel^0 = 1 + \eta n_\parallel^0 \hat{a} \) is a constant unit vector, and

\[
\left\langle f(p_1, p_2, p_3, p_4) \right\rangle = \int dp_1 dp_2 dp_3 dp_4
\]

\[
\times \delta(p_1 + p_2 - p_3 - p_4)
\]

\[
\times \delta(e_1 + e_2 - e_3 - e_4)
\]

\[
\times f(p_1, p_2, p_3, p_4).
\]

Combining Eq. (4) with the continuity equation of the spin density \( \mathbf{m}(r, \tau, \theta) \), which is readily derived by integrating Eq. (1) over \( p_i \), we obtain the macroscopic equation for the spin density,

\[
\frac{\partial \mathbf{m}}{\partial \tau} - \mathbf{m} \times \Omega_0 = \sum_i \frac{\partial}{\partial r_i} \left[ D_\parallel \frac{\partial \mathbf{m}}{\partial r_i} \hat{\mathbf{e}} + \frac{D_\perp}{1 + \mu^2 m^2} \times \left( \frac{m \frac{\partial \mathbf{e}}{\partial r_i} - \eta \hat{\mathbf{e}} \times \frac{\partial \mathbf{e}}{\partial r_i} \right) \right].
\]

(7)

When \( D_\parallel = D_\perp \), Eq. (7) reduces to that given in Refs. 1 and 2. Note that Eq. (7) reduces to the conventional Fick’s law form when \( \mathbf{m} \) is parallel to the external field \( \hat{\mathbf{e}} \). When \( \mathbf{m} \) is parallel to the static external field \( \hat{\mathbf{e}} \), we can linearize Eq. (7) in \( \mathbf{m} \) to give

\[
\frac{\partial \delta \mathbf{m}}{\partial \tau} - \delta \mathbf{m} \times \Omega_0 = D_\parallel \nabla^2 \delta \mathbf{m}_\parallel + D_\perp (1 + \mu^2 m_0^2)^{-1}
\]

\[
\times \left( \nabla^2 \delta \mathbf{m}_\perp - \eta \hat{\mathbf{e}} \times \nabla^2 \delta \mathbf{m}_\parallel \right),
\]

(8)

where \( \delta \mathbf{m}_\parallel \) (\( \delta \mathbf{m}_\perp \)) is the component of \( \mathbf{m} \) parallel (perpendicular) to \( \mathbf{m}_0 \). Equation (8) for fermions is the same as that originally presented by Meyerovich.11

In the classical regime where particles obey Boltzmann statistics, \( \tau_\parallel, \tau_\perp \) and \( D_\parallel, D_\perp \) are the same for Fermi and Bose gases having the same density and polarization. This can be readily seen from Eqs. (5) and (6) and the expressions for \( D_\perp \) for both \( \tau_\parallel = \tau_\perp \) and \( D_\parallel = D_\perp \). It is also not hard to show that \( \tau_\parallel = \tau_\perp \) and \( D_\parallel = D_\perp \), and both \( \tau \) and \( D \) are independent of polarization in the classical regime. In this limit we obtain the analytic forms

\[
\tau_\parallel = \tau_\perp = (3/32n \sqrt{\pi})(k_B T/m^*)^{-1/2},
\]

(9)

\[
D_\parallel = D_\perp = (3/32n \sqrt{\pi})(k_B T/m^*)^{1/2},
\]

(10)

where \( k_B \) is the Boltzmann constant. These are the same as the s-wave values of Ref. 2. Using arguments presented in Ref. 12, it can be shown that, for \( M \rightarrow 0 \), \( \tau_\parallel = \tau_\perp \) and \( D_\parallel = D_\perp \) for all temperatures above \( T_c^+ \),

\[
T_c^+ = (\hbar^2/2 \pi m^* k_B)(n_/\zeta(1/2))^{2/3},
\]

where the Riemann \( \zeta \) function \( \zeta(1/2) \approx 2.612 \).
We evaluate \( \tau_\parallel, \tau_\perp, D_\parallel, \) and \( D_\perp \) for a Bose gas following the methods used in the fermion case.\(^{12} \) Since \( \mu_0 < 0 \) for bosons, \( L_\parallel \) and \( L_\perp \) can always be expressed as fourfold summations similar to that appearing in fermion case. We evaluate these series numerically. The results for both \( \tau_{\parallel,\perp} \) and \( D_{\parallel,\perp} \) are shown in Figs. 1 and 2, respectively, where we see that \( \tau_\parallel \neq \tau_\perp \) and both depart from the classical limit (9) at low \( T \).

Perhaps the most remarkable feature in Fig. 1 is the decrease of \( \tau_\parallel \) with decreasing \( T \) for \( T < 1.6T_c + \). Since the dispersion relation for transverse spin waves is given by

\[
\omega = \Omega_0 + D_\perp k^2/(i - \eta \mu m_0),
\]

where \( k \) is the wave vector [cf. Eq. (8)], this means the intrinsic damping of spin waves will increase as \( T \) decreases in this temperature regime. As \( T \) decreases, \( \tau_\parallel \) continues to increase for \( M > 0.51 \) but for \( M < 0.51 \) it reaches a maximum and decreases. \( \tau_\parallel \) breaks away from \( \tau_\perp \) at higher temperatures at larger \( M \). We can show that, for \( T = T_c + \),

\[
f(k, l, m, n) = (\sqrt{\pi}/8)(k + l + m + n + 2)^{3/2}/[(k + l + 1)(m + n + 1)(l + m + 1)]^2(k + n + 1^2).
\]

In Fig. 2, we see that \( D_\parallel \) goes to 0 for all \( M \) as \( T \to T_c + \). As noted above, the mean-field contribution to the drift term gives a nonvanishing \( D_\parallel \) for \( T \to T_c + \). The proper inclusion of these interaction effects modifies \( \alpha_\parallel (\text{not } \tau_\parallel) \) and \( \alpha_\perp \) becomes (instead of going to 0)

\[
\alpha_\parallel \sim (m^* / k_B T_\epsilon)(a / r_0)
\]

for \( T \to T_c + \). Here \( r_0 \) is the average particle spacing. Since \( a / r_0 \ll 1 \) for dilute gases, \( D_\parallel \) for \( T \to T_c + \) is still very small. For negative \( a \) (\( a > 0 \) for \( H \)), \( D_\parallel \) has a complicated structure near \( T_c + \), which we cannot discuss in this short work. For \( D_\perp \), interaction effects are negligible except for very small polarization (\( M \leq a / r_0 \)) for \( T \to T_c + \). For this case \( D_\perp \approx D_\parallel \). For \( M > a / r_0 \), interaction effects on \( D_\perp \) are always negligible. One can show that

\[
\tau_\parallel = \tau_\parallel = (\sqrt{\pi}/24)\xi(\xi^2/4)S_1^{-1} A = 0.013 A, \quad \text{for } M \to 0,
\]

\[
\tau_\parallel = [(\sqrt{\pi}/3) / 2^{10/3}]S_2^{-1} A = 0.030 A, \quad \text{for } M \to 1
\]

\[
\tau_\parallel = [(\sqrt{\pi}/3) / 2^{10/3}]\xi(\xi^2/4)S_3^{-1} A = 0.016 A, \quad \text{for } M \to 1
\]

where

\[
S_1 = \sum_{k, l, m, n} f(k, l, m, n) = 14.8,
\]

\[
S_2 = \sum f(k, l, m, n) \delta_{m,0} \delta_{n,0} = 1.98,
\]

\[
S_3 = \sum f(k, l, m, n) \delta_{n,0} = 4.9,
\]

and

\[
A = [3/(8\pi k_B T_c)] h^3 / m^* .
\]

The Riemann \( \xi \) function \( \xi(\xi^2/4) \approx 1.341 \) and

\[
T_c = (h^2 / 2\pi m^* k_B)[(n / 2\xi(\xi^2/4))^2]^{1/3}
\]

is the critical temperature for the unpolarized system and

\[
D_\parallel = (\sqrt{\pi}/3)^{11/3}[(\xi(\xi^2/4))/2^{10/3}]S_3^{-1}B \approx 6.6 \times 10^{-3}B
\]

for \( M \to 1 \) and \( T \to T_c + \), where \( B = 2ak_B T_c / m^* \).

This study was motivated by a recent experiment\(^9 \) on \( H^1 \) in which an anomalous increase in damping of standing spin-wave modes at low \( T \) was reported. This was explained in terms of surface-dephasing effects.\(^9 \) Since the anomalous damping occurred at \( T \sim 100T_c + \), the reported density, it is unlikely that our result of an intrinsic increase in damping near \( T_c + \) due to degeneracy is responsible for this anomalous damping. We leave the study of the spin dynamics for the Bose condensed regime \( (T < T_c +) \) for future work.

We thank Professor W. J. Mullin for his interest and many helpful discussions and thank Dr. P. Findenisen for help with numerical calculations. Support from Natural Sciences and Engineering Research Council (NSERC), Canada is gratefully acknowledged.

14V. P. Silin, Vvedenie v Kineticheskuyu Teoriyu Gasov (Nauka, Moscow, 1971), Chap. 8.
16For negative $M$, $\tau_{\perp,\perp}(M) = \tau_{\perp,\perp}(|M|)$ and $D_{\perp,\perp}(M) = D_{\perp,\perp}(|M|)$.
19$\Delta g(r,t)$ and $\tau(k_0,k_f)$ here are the same as $\hbar^2P$ and $(2\pi)^2T(k_0,k_f)$ in Ref. 2.
21As discussed in Ref. 1, the small force in the drift term arising from the inhomogeneity of the external field is negligible in usual experimental situations.