

Chapter 8

Fermi Systems

8.1 The Perfect Fermi Gas

In this chapter, we study a gas of non-interacting, elementary Fermi particles. Since the particles are non-interacting, the potential energy is zero, and the energy of each Fermion is simply related to its momentum by $\epsilon = \frac{p^2}{2m}$. A Fermion has one-half integral spin, which we denote by s . The state of the Fermion depends on the “orientation” of this spin (with respect to an applied magnetic field) as well as on its location in phase space. For spin s , there are $2s + 1$ spin states or “orientations” of s . In the absence of a magnetic field, these $2s + 1$ states all have the same energy and serve simply to expand the number of states available to the Fermion. Thus the density of states in volume element $d\Gamma$ of phase space, including the spin states, is $g_s \frac{d\Gamma}{h^3}$ where $g_s = (2s + 1)$. The statistical mechanics of the Fermi gas follows directly from the Grand Canonical Partition function (5.24) and the Fermi function

$$\bar{n}(\epsilon) = \left(e^{\beta(\epsilon - \mu)} + 1 \right)^{-1} \quad (8.1)$$

which gives the expected number of Fermions in energy state ϵ . The expected number of Fermions in energy range ϵ to $\epsilon + d\epsilon$ is then $dN = \bar{n}_s g_s \frac{d\Gamma}{h^3}$ or

$$dN(\epsilon) = \bar{n}(\epsilon) g_s g(\epsilon) d\epsilon$$

where, as in (4.7), $g(\epsilon) = 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} \epsilon^{1/2}$.

We consider specifically a fixed number N of Fermions in volume V having number density $n = N/V$. Then $N = \sum_s \bar{n}_s = g_s \int \bar{n} \frac{d\Gamma}{h^3}$ or

$$N = g_s 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{\left(e^{\beta(\epsilon - \mu)} + 1 \right)}. \quad (8.2)$$

The integral here cannot be further reduced because we don't know the chemical potential $\mu(T)$. However, since N is fixed and the factors outside the integral

are constant, we do know that $\mu(T)$ must vary with T so as to keep the integral constant. In fact, we will use the relation to determine $\mu(T)$. We shall see that at $T = 0$ K μ is equal to the Fermi energy, ϵ_F , and as T is increased μ decreases uniformly until it joins on to its classical limit $\mu = -kT \log(Z_0/N)$ at very high temperature. Similarly, the internal energy is $U = \sum_s \bar{n}_s g_s = g_s \int \bar{n}(\epsilon) g(\epsilon) d\epsilon$ or

$$U = g_s 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{(e^{\beta(\epsilon-\mu)} + 1)}. \quad (8.3)$$

and we see that the chemical potential plays a central role in determining the properties of a Fermi gas.

We now specialize to spin 1/2 Fermions for which $g_s = 2$. In the presence of a magnetic field (say along the z axis) the spin now aligns either parallel or anti-parallel to the magnetic field, $s_z = \pm 1/2$, to make up the two possible spin states. Our gas is then a model of conduction electrons in a metal or of liquid ^3He .

In the case of conduction electrons, the model neglects the interaction between the electrons and the interaction of the electrons with the ions in the metal. The electron-electron interaction is simply the Coulomb interaction $v(r) = e^2/r$. This varies very slowly with r and has a very long range so that each electron interacts with many other electrons. If the electron density is high, a given electron interacts with so many electrons that the total potential energy it experiences is almost flat. The potential energy is then almost constant and can, to a good approximation, be subtracted off as a constant.

The model also neglects the interaction between the conduction electrons and the ions in the metal. This electron-ion interaction is a sensitive balance between the electron-nucleus attraction and the electron-ion core electron repulsion. In suitable cases, these two interactions largely cancel leaving the conduction electrons effectively free. Examples where this cancellation is nearly complete are the alkali metals and aluminum, and in these cases, a Fermi gas is a good model of the conduction electrons.

In the case of liquid ^3He , we simply ignore the ^3He - ^3He atom interactions.

8.2 Zero Temperature

At $T = 0$ K, the Fermi gas will be in its ground state. This is formed by filling up the lowest energy states until all the N Fermions are accommodated. Since there are two spin states per space state, this requires $N/2$ space states in phase space. The configuration space part of phase space is just the volume V . Thus, we must fill up a sphere in momentum space of volume $4\pi p_F^3/3$ such that

$$\frac{1}{h^3} \left(V \frac{4\pi p_F^3}{3} \right) = \frac{N}{2} \quad (8.4)$$

where h^3 is the volume of phase space taken up by one state. The momentum p_F of the most energetic Fermion is called the Fermi momentum. We could

picture this physically using our Particle Hilton discussed in Section 5.3. The Fermion Hilton has two rooms per floor, one must be occupied by a spin up Fermion ($s_z = 1/2$) and the other by a spin down Fermion ($s_z = -1/2$). To form the ground state in the Hilton, we fill up the lowest $N/2$ floors with one Fermion per room. This “sea” of occupied states is often called the Fermi sea.

The result (8.4) may be obtained more formally using $N = \sum_s \bar{n}_s$ where $\bar{n}_s = 1$ up to the last occupied state and $\bar{n}_s = 0$ beyond, i.e.,

$$\begin{aligned} N &= \sum_s 1 = 2 \int \frac{d\Gamma}{h^3} \times 1 \\ &= 2V \frac{4\pi}{h^3} \int_0^{p_F} p^2 dp = \frac{2}{h^3} \left(V \frac{4\pi p_F^3}{3} \right). \end{aligned}$$

The Fermi momentum is, therefore, simply related to the particle density n by $p_F = \hbar (3\pi^2 n)^{1/3}$.

The higher the density (the smaller the volume V), the larger the sphere in momentum space required to occupy all the Fermions. The corresponding energy of the most energetic Fermion is

$$\epsilon_F = \frac{p_F^2}{2m} = \left(\frac{\hbar^2}{2m} \right) (3\pi^2 n)^{2/3}, \quad (8.5)$$

called the Fermi Energy. This could also have been obtained directly as

$$\begin{aligned} N &= \sum_s \bar{n}_s = 2 \int_0^{\epsilon_F} d\epsilon g(\epsilon) \times 1 \\ &= 2(2\pi V) \left(\frac{2m}{\hbar^2} \right)^{3/2} \frac{2}{3} \epsilon_F^{3/2} \end{aligned} \quad (8.6)$$

Comparison of eqs. (8.2) and (8.6) suggests a convenient expression for $g(\epsilon)$ in terms of ϵ_F as,

$$g(\epsilon) = \frac{3}{4} \frac{N}{\epsilon_F} \left(\frac{\epsilon}{\epsilon_F} \right)^{1/2} \quad (8.7)$$

It is also convenient to define a density of states including directly both spin states

$$g_B(\epsilon) \equiv 2g(\epsilon) = \frac{3}{2} \frac{N}{\epsilon_F} \left(\frac{\epsilon}{\epsilon_F} \right)^{1/2} \quad (8.8)$$

To locate the chemical potential, we recall that at $T = 0$ K the expected occupation is $\bar{n}(\epsilon) = 1$ for states of energy $\epsilon < \epsilon_F$ and all higher states are empty. Hence,

$$\bar{n}(\epsilon) = \frac{1}{(e^{\beta(\epsilon-\mu)} + 1)} = \begin{cases} 1 & \epsilon < \epsilon_F \\ 0 & \epsilon > \epsilon_F \end{cases} \quad (8.9)$$

Since $\beta \rightarrow \infty$ at $T = 0$ K, the Fermi function can fulfill this condition only if $\mu = \epsilon_F$. Hence, at $T = 0$ K μ , is equal to the Fermi energy. This is consistent with the thermodynamic relation $\mu = \left(\frac{\partial U}{\partial N}\right)_{N=\bar{N}}$ for to add an additional Fermion (beyond the fixed number \bar{N}), we must place this particle at energy ϵ_F since all lower states are occupied.

At $T = 0$ K, the statistical properties of the Fermi gas are given simply by eq. (8.9). This distribution is displayed in Fig. 7.1 along with the distribution of Fermions over energy, $\bar{n}(\epsilon)g(\epsilon)$. When all the lowest states are occupied as depicted in Fig. 7.1, the Fermi gas is said to be degenerate. The thermodynamic properties can be calculated from the internal energy

$$U = \int_0^{\epsilon_F} d\epsilon g_B(\epsilon)\epsilon = \frac{3}{5}N\epsilon_F$$

which gives an average energy per particle of

$$\bar{\epsilon} = \frac{U}{N} = \frac{3}{5}\epsilon_F \quad (8.10)$$

The pressure exerted by the Fermi gas is, for example,

$$p = -\left(\frac{\partial U}{\partial V}\right)_s = N\frac{2}{5}\epsilon_F = \frac{2}{3}\frac{U}{V} \quad (8.11)$$

This gives a Gibbs Free energy of

$$G = U + pV = N\epsilon_F \quad (8.12)$$

which, from the thermodynamic relation $G = \mu N$, verifies that $\mu = \epsilon_F$ at $T = 0$ K. Other properties follow from differentiating U .

To complete the $T = 0$ K study, we seek the temperature at which a Fermi gas can be approximated by a degenerate, $T = 0$ K, Fermi gas. This is the Fermi temperature $T_F = \frac{\epsilon_F}{k}$ which sets the temperature scale. For $T \ll T_F$ the temperature is essentially zero, and the Fermi gas is degenerate. The Fermi temperature for conduction electrons in a metal is, from (8.5),

$$T_F = \left(\frac{\hbar^2}{2mk}\right) (3\pi^2 n)^{2/3} = 4.26 \times 10^5 n^{2/3} \text{K} \quad (8.13)$$

where in the last equality n is in units of electrons/ \AA^3 . Table 7.1 lists some typical values of T_F , and we see that room temperature is a low temperature ($T \ll T_F$) for electrons in metals. By contrast, the Fermi temperature for liquid ^3He atoms, regarded as non-interacting, is from eq. (8.13) just 5K . For liquid ^3He to be nearly degenerate, temperatures below 100 mK are required.

Metal	a_0 (\AA)	n (\AA^{-3})	T_F (K)	Θ_D (K)	C_V/R (K)
Cu	3.61	0.0850	82,000	315	.018
Li	3.46	0.0483	56,000	315	.026
K	5.25	0.0138	24,000	100	.062
Al	4.04	0.1820	137,000	400	.011

Table 8.1: The electron density is related to the lattice constant by $n = Z(\frac{N}{a_0^3})$, where Z the number of conduction electrons per atom, and N is the number of atoms per unit cell of volume a_0^3 . C_V/R is the electronic specific heat at room temperature assuming $m^* = m$. Θ_D is the Debye temperature.

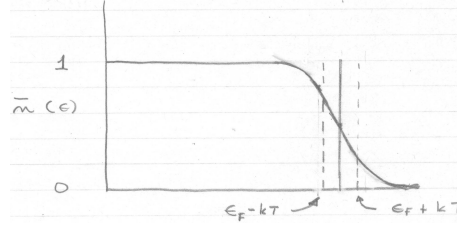


Figure 8.1

8.3 Low Temperature

At low temperature, $T \ll T_F$, the particle occupation will differ only slightly from the $T = 0$ K completely degenerate occupation. The $\bar{n}(\epsilon)$ for $T \ll T_F$ is shown in Fig. 8.2. At temperature T , the electrons have thermal energy kT . Only those electrons having an energy state within kT of the Fermi surface, i.e. immediately below the Fermi surface ($\epsilon = \mu$) can be excited. The resulting change in $\bar{n}(\epsilon)$ can be estimated by substituting $\epsilon = \mu \pm kT$ in the Fermi function (8.1)

$$\begin{aligned} \epsilon = \mu + kT ; \quad \bar{n}(\epsilon) &= \frac{1}{e+1} \approx 0.28 \\ \epsilon = \mu - kT ; \quad \bar{n}(\epsilon) &= \frac{1}{e^{-1}+1} \approx 0.79 \end{aligned} \quad (8.14)$$

Thus, only the $\bar{n}(\epsilon)$ at energies ϵ within kT of the Fermi surface is modified at low temperature.

8.3.1 Approximate C_V

To excite the Fermions to higher states above ϵ_F requires energy which leads to a specific heat. The measurement of the specific heat is, therefore, a test of the Fermi gas model. To estimate C_V , we note that the number of Fermions excited when the system is heated from temperature T is approximately,

$$\delta N = \left(\frac{\partial N}{\partial \epsilon} \right)_{\epsilon_F} \delta \epsilon \quad (8.15)$$

where $(\partial N/\partial \epsilon)_{\epsilon_F} = g_B(\epsilon_F)$ is the density of states at the Fermi surface and $\delta \epsilon \sim kT$ is, from (8.14), the region of energy affected at temperature kT . Also, the excitation energy of each Fermion is roughly the classical thermal energy. Thus, the energy required to heat the Fermi gas to temperature kT (the thermal energy) is

$$\delta U \sim \delta N kT = g_B(\epsilon_F) (kT)^2 \quad (8.16)$$

This gives a specific heat of

$$C_V = \left(\frac{\delta U}{\delta T} \right)_V = 2g_B(\epsilon_F) k^2 T \quad (8.17)$$

Substituting for $g_B(\epsilon_F)$, this is

$$C_V = 3(kN) \frac{T}{T_F} = 3R \frac{T}{T_F} \quad (8.18)$$

per mole of Fermions. The “exact” result derived below is

$$C_V = k \frac{\pi^2}{3} g_B(\epsilon_F) kT = \frac{\pi^2}{2} R \frac{T}{T_F} \quad (8.19)$$

per mole.

The Fermi gas specific heat is roughly the density of states at the Fermi surface times the thermal energy. The higher the density of states, the larger the number of electrons excited, and the greater is C_V . The C_V is substantially lower than the classical value of $3/2R$ per mole because only the fraction of the electrons that are within kT of ϵ_F are excited.

The result (8.19) applies directly to electrons in metals with $g_B(\epsilon_F)$ given by the density of states of electrons in the metal taking account of the electron-ion interactions. This metallic density of states can be either greater or less than the free electron value (8.7) depending on the detailed nature of the states available to the electrons in the metal (the band structure). Measurement of C_V , therefore, yields direct information on the density of states. Since $g_B(\epsilon_F) \propto m$, the Free electron C_V is often fitted to the observed C_V to obtain an effective mass m^* for the electron in a metal. The m^* represents the band structure arising from the interaction with the ions and the other electrons, i.e.,

$$C_V = \left(\frac{m^*}{m} \right) k \frac{\pi^2}{3} g_B(\epsilon_F) kT \quad (8.20)$$

If the ratio of the effective mass to the “bare” mass, m^*/m , is unity then the electrons are nearly “free.” In Li $m^*/m = 2.4$ while in K $m^*/m = 1.1$. For ^3He atoms in liquid ^3He $m^*/m = 3.1$ and this effective mass can be directly related to the parameters proposed by Landau to describe the ^3He - ^3He interaction in the liquid.

8.3.2 Accurate C_V

To obtain an accurate expression for C_V , we first determine how the chemical potential, $\mu(T)$, varies with T from the fact that

$$N = \int_0^{\infty} \bar{n}(\epsilon) g(\epsilon) d\epsilon \quad (8.21)$$

must be constant independent of T and then calculate the energy

$$U = \int_0^{\infty} \bar{n}(\epsilon) g(\epsilon) \epsilon d\epsilon \quad (8.22)$$

The specific heat is then obtained as $C_V = dU/dT$. To do this, we must evaluate integrals over the Fermi function $n(\epsilon)$ of the form

$$\begin{aligned} I &= \int_0^{\infty} \bar{n}(\epsilon) f(\epsilon) d\epsilon \\ &= \bar{n}(\epsilon) F(\epsilon) \Big|_0^{\infty} - \int \left(\frac{d\bar{n}(\epsilon)}{d\epsilon} \right) F(\epsilon) d\epsilon \end{aligned} \quad (8.23)$$

where $F(\epsilon) = \int_0^{\epsilon} f(\epsilon') d\epsilon'$. Here $f(\epsilon)$ is any function of energy but it will always contain the density of states (i.e., $f(\epsilon) = g(\epsilon)$ in (8.21) and $f(\epsilon) = g(\epsilon)\epsilon$ in (8.22)), so $F(0) = 0$. At $\epsilon = \infty$, $n(\epsilon) = 0$ so the integrated term in (8.23) vanishes. With this integration by parts, we have

$$I = \int_0^{\infty} d\epsilon \left(-\frac{d\bar{n}}{d\epsilon} \right) F(\epsilon) d\epsilon \quad (8.24)$$

To evaluate I at low temperature ($kT \ll \epsilon_F$), we exploit the property that the Fermi function is nearly a step function at low T . Thus $(-d\bar{n}/d\epsilon)$ is nearly a Delta function,

$$-\frac{d\bar{n}}{d\epsilon} \simeq \delta(\epsilon - \mu)$$

peaked at the edge of the Fermi function, at the Fermi surface $\epsilon = \mu$. We will then need $F(\epsilon)$ accurately only in the energy range $\epsilon \simeq \mu$. Provided $F(\epsilon)$ is a smoothly varying function of ϵ , we may expand $F(\epsilon)$ in a Taylor series about $\epsilon - \mu$ and use this series in (8.24), i.e.,

$$F(\epsilon) = F(\mu) + \sum_{n=1}^{\infty} \frac{1}{n!} F^{(n)}(\mu) (\epsilon - \mu)^n$$

and

$$I = \int_0^{\infty} d\epsilon \left(-\frac{d\bar{n}}{d\epsilon} \right) \left[f(\mu) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(\mu) (\epsilon - \mu)^n \right]$$

where $F^n(\mu) = (d^n F/d\epsilon^n)_{\epsilon=\mu}$. From the definition of $F(\epsilon)$, we have $F^n(\mu) = f^{n-1}(\mu)$. Introducing the moments of the Delta function $(-d\bar{n}/d\epsilon)$,

$$\begin{aligned} M_n &= \frac{1}{n!} \int_0^\infty d\epsilon \left(-\frac{d\bar{n}}{d\epsilon} \right) (\epsilon - \mu)^n \\ &= \frac{(kT)^n}{n!} \int_{-\beta\mu}^\infty dx \left(-\frac{d\bar{n}}{dx} \right) x^n \end{aligned}$$

where $x = \beta(\epsilon - \mu)$, we have

$$I = F(\mu) M_0 + \sum_{n=1}^{\infty} f^{n-1}(\mu) M_n \quad (8.25)$$

In this way, we obtain I as a power series in $(kT)^n$ valid at low temperature T involving the derivatives of $f(\epsilon)$ and moments, M_n , of the Delta function $(-d\bar{n}/d\epsilon)$. Since $(-d\bar{n}/d\epsilon) \simeq \delta(\epsilon - \mu)$ is an even function, and we may extend $-\beta\mu$ to $-\infty$, all the odd moments M_n vanish. The first three even moments are ¹

$$M_0 = 1 \quad M_2 = \frac{(\pi kT)^2}{6} \quad M_4 = \frac{4}{360} (\pi kT)^4$$

Also, since $\mu - \epsilon_F$ is small, we have

$$F(\mu) \simeq \int_0^{\epsilon_F} d\epsilon f(\epsilon) + (\mu - \epsilon_F) f(\epsilon_F)$$

and

$$I = \int_0^{\epsilon_F} d\epsilon f(\epsilon) \left[(\mu - \epsilon_F) f(\epsilon_F) + \frac{(\pi kT)^2}{6} f'(\epsilon_F) + \frac{7}{360} (\pi kT)^4 f'''(\epsilon_F) + \dots \right]. \quad (8.26)$$

¹The second moment M_2 is evaluated using

$$\begin{aligned} \frac{d\bar{n}}{d\epsilon} &= \frac{d}{d\epsilon} \left[e^{\beta(\epsilon-\mu)} + 1 \right]^{-1} \\ &= -\beta \frac{e^{\beta(\epsilon-\mu)}}{[e^{\beta(\epsilon-\mu)} + 1]^2} \end{aligned}$$

so that

$$M_2 = \frac{1}{2} \int_0^\infty d(\beta\epsilon) \frac{e^{\beta(\epsilon-\mu)}}{[e^{\beta(\epsilon-\mu)} + 1]^2} (\epsilon - \mu)^2$$

with $x = \beta(\epsilon - \mu)$, this can be written as,

$$M_2 = \frac{(kT)^2}{2} \int_{-\beta\mu}^\infty dx \frac{e^x}{(e^x + 1)^2} x^2 = \frac{\pi^2}{6} (kT)^2$$

where the final result is obtained letting $\beta\mu \rightarrow \infty$.

Keeping terms up to $(kT)^2$ only, the N is (8.21) ($f = g_B(\epsilon)$) and U in (8.22) ($f = g_B(\epsilon)\epsilon$), where $g_B(\epsilon) = (3N/2\epsilon_F)(\epsilon/\epsilon_F)^{1/2}$ is the density of states for both spins are,

$$N = \int_0^{\epsilon_F} g_B(\epsilon) d\epsilon + \left[(\mu - \epsilon_F)g_B(\epsilon_F) + \frac{\pi^2}{6}(kT)^2 g'_B(\mu) \right] + \dots \quad (8.27)$$

and

$$U = \int_0^{\epsilon_F} g_B(\epsilon) \epsilon d\epsilon + \epsilon_F \left[(\mu - \epsilon_F)g_B(\epsilon_F) + \frac{\pi^2}{6}(kT)^2 g'_B(\mu) \right] + \frac{\pi^2}{6}(kT)^2 g_B(\epsilon) + \dots \quad (8.28)$$

Since the first term in (8.27) is the value of N at $T = 0$ K, the square bracket in (8.27) vanishes. Setting $\mu = \epsilon_F$ in the $(kT)^2$ term in the square bracket gives an expression for μ of

$$\mu = \epsilon_F \left[1 - \frac{1}{12} \left(\frac{\pi kT}{\epsilon_F} \right)^2 \right] \quad (8.29)$$

Similarly, the square bracket in U vanishes so that

$$\begin{aligned} U &= U_0 + \frac{\pi^2}{6}(kT)^2 g_B(\epsilon_F) \\ &= \frac{3}{5} N \epsilon_F \left[1 + \frac{5}{12} \left(\frac{\pi kT}{\epsilon_F} \right)^2 \right] \end{aligned} \quad (8.30)$$

Differentiation of U leads directly to the specific heat given in (8.19).

We see also from (8.29) that μ decreases with increasing T . This trend is confirmed by higher order terms in the expansion (the second order term is $-7/960(\frac{\pi kT}{\epsilon_F})^4$) and μ goes over to its classical value $\mu = -kT \log(Z_0/N)$ at $T \geq T_F$.

The values of the electronic specific heat for some typical metals at room temperature are listed in Table 7.1. There we see that C_V is typically $\sim 1 - 2\%$ of classical Dulong-Petit value ($\frac{3}{2}R$). The electronic degrees of freedom are largely “frozen out” at room temperature because it is not possible to excite the majority of electrons buried deep in the Fermi sea. This explanation of why C_V of electrons lies so far below the classical value, proposed by Sommerfield, was another triumph of quantum mechanics. Thus, at room temperature, the electronic specific heat is negligible compared to the phonon specific heat.

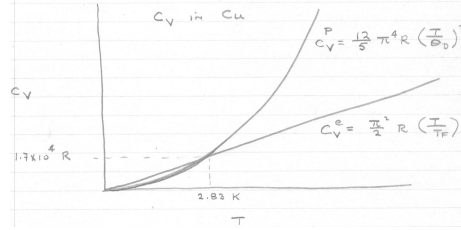


Figure 8.2

Since C_V^e in (8.19) is linear in T and the phonon specific heat C_V^P goes as T^3 , at low enough temperature C_V^e must dominate. To find the temperature at which they are equal, we solve for T in $C_V^P = C_V^e$ or $\frac{12}{5}\pi^4 R \left(\frac{T}{\Theta_D}\right)^3 = \frac{\pi^2}{2} R \left(\frac{T}{T_F}\right)$

which gives $T = \left(\frac{5}{2\pi^2} \frac{\Theta_D^3}{T_F}\right)^{1/2}$.

For Cu , for example, $T = 2.83K$, and it is only at very low temperature when the phonons become “frozen out” that the electronic specific heat becomes important.

At this value $C_V^{\text{phonons}} = C_V^{\text{electrons}} = 1.69 \times 10^{-4}R$.

8.4 Pauli Paramagnetism

We noted in Section 8.1 that spin 1/2 Fermions had two intrinsic states for each space state. Uhlenbeck and Goudsmit in 1925 first discovered this property for electrons. While the intrinsic states had the same energy in the absence of a magnetic field, they found that in the presence of a magnetic field, the electron displays an apparent intrinsic magnetic moment

$$\vec{\mu} = \left(\frac{e}{m}\right) \vec{s} \quad (8.31)$$

The general expression valid for all particles is $\vec{\mu} = g\left(\frac{e\hbar}{2m}\right)\vec{s}$ where g is the Landé g factor, and for an electron, the charge e is negative. For the electron $g = 2.0023 \approx 2$. The two states corresponded to the spin s taking the values $s_Z = +\frac{1}{2}\hbar$ or $s_Z = -\frac{1}{2}\hbar$ where the z axis is defined as parallel to the applied magnetic field B . The energy of the electron is then

$$E = -\vec{\mu} \cdot \vec{B} = \pm \left(\frac{e\hbar}{2m}\right) B = \pm \mu_B B \quad (8.32)$$

where $\mu_B = (e\hbar/2m)$ is the Bohn magneton. In this way, the energy of the two intrinsic states differ in a magnetic field. Since the states manifest themselves in a magnetic field in the same form as the angular momentum, the states were denoted as the intrinsic spin states of the particle. Spin was introduced into quantum mechanics by Pauli.

If we suppose the energy of the electron was ϵ , the electrons in the two spins states split to $\epsilon_{\uparrow} = \epsilon - \mu_B B$ and $\epsilon_{\downarrow} = \epsilon + \mu_B B$. Electrons will transfer from

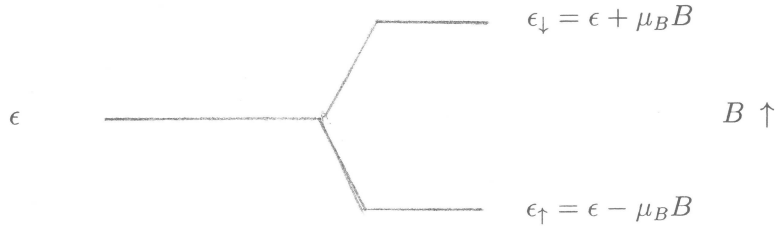


Figure 8.3

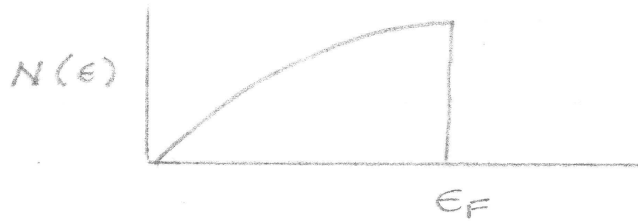


Figure 8.4

the higher energy state ϵ_{\downarrow} to the lower energy state in order to minimize the total energy (Fig. 8.3).

The spin magnetic moments will tend to align with the field B . We now set out to find how many will align with the field and what magnetization is induced. In Section 7.2, we saw that the $\bar{n}(\epsilon)$ at $T = 0$ K was a step function (Fig. 7.1) and that, including the density of states $g(\epsilon)$ in (8.7), the number of Fermions having energy ϵ is $N(\epsilon) = \bar{n}(\epsilon)g(\epsilon) \propto \epsilon^{1/2}$. This distribution is depicted in Fig. 8.4.

In the absence of an applied magnetic field ($B = 0$), the energy of the two spin states is the same, $\epsilon_{\uparrow} = \epsilon_{\downarrow} = \epsilon_F$, as shown on the LHS of Fig. 8.5. The two spin states have the same chemical potential $\mu = \epsilon_F$. In the presence of a field B , the energy of spin state ϵ_{\downarrow} moves above ϵ_{\uparrow} , their chemical potentials differ, as shown in the middle of Fig. 8.5. The gas is not in equilibrium at this moment since in equilibrium the chemical potential of all the Fermions. Fermions will ??? over from state ϵ_{\downarrow} to ϵ_{\uparrow} (flip their spins) to equate the chemical potential of ϵ_{\uparrow} and ϵ_{\downarrow} .

The number δN of particles that flip their spins to equate ϵ_{\uparrow} and ϵ_{\downarrow} is

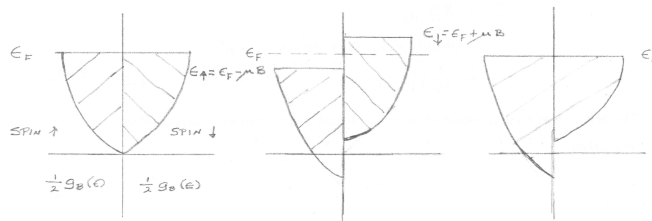


Figure 8.5

$$\delta N_\epsilon = \frac{dN}{d\epsilon} \Big|_{\epsilon=\mu} \delta\epsilon = \frac{1}{2} g_B(\epsilon_F) \mu_B H \quad (8.33)$$

where $g_B(\epsilon_F) = \frac{3N}{2\epsilon_F}$ is the density of states at $\epsilon = \epsilon_F$ as before and $\delta\epsilon = \mu_B H$.

This is the number of particles δN that change their spin orientation from $s = -\frac{1}{2}$ to $s = +\frac{1}{2}$. As a result, the magnetic moment induced is

$$M = 2\mu_B \delta N = g_B(\epsilon_F(0)) \mu_B^2 B \quad (8.34)$$

The magnetic moment per unit volume (the magnetization) is

$$m = \frac{M}{V} = \frac{g_B(\epsilon_F)}{V} \mu_B^2 B \quad (8.35)$$

Thus the paramagnetic susceptibility is

$$\begin{aligned} \chi_P \equiv \frac{m}{B} &= \frac{g(\epsilon_F)}{V} \mu_B^2 \\ &= \text{density of states per unit volume} \times \mu_B^2 \\ &= \frac{3}{2\epsilon_F} \left(\frac{N}{V} \right) \mu_B^2 \end{aligned} \quad (8.36)$$

This is known as Pauli paramagnetism to distinguish it from temperature dependent magnetism.