1. Consider \(N\) non-interacting Bosons in a harmonic magnetic trap. The potential energy seen by each independent Boson is

\[ V(x, y, z) = \frac{1}{2} n \left[ \omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2 \right] \]

and the energy states of the Bosons is

\[ \epsilon(n_1, n_2, n_3) = (n_1 + \frac{1}{2}) \hbar \omega_1 + (n_2 + \frac{1}{2}) \hbar \omega_2 + (n_3 + \frac{1}{2}) \hbar \omega_3 \]

where \(n_1, n_2\) and \(n_3\) are integers. The density of states of the Bosons is,

\[ g(\epsilon) = \frac{\epsilon^2}{2\hbar^3 \omega_1 \omega_2 \omega_3} \]

(a) Show following the method used for a uniform Bose gas that the condensate fraction for temperatures \(T\) below \(T_C\) is

\[ \frac{N_0}{N} = 1 - \left( \frac{T}{T_C} \right)^3 \]

(b) In a 2D trap, the DOS is \(g(\epsilon) = \epsilon / \hbar^2 \omega_1 \omega_2\). Determine \(N_0/N\) for this case.

(c) In the 3D case, for Rb atoms (\(m = 87\) Amu) and \(N = 10^{12}\) atoms in the trap and \(\omega_1 = \omega_2 = \omega_3 = 2\pi 60\) Hz, estimate the BEC condensation temperature \(T_C\). To obtain \(T_C\), \(g_\nu(z=1)\) will have to be re-evaluated using the DOS for atoms in a trap.

2. Using the expression,

\[ g(\epsilon) = \sum_s \delta(\epsilon - \epsilon_s) \]

\[ = \sum_{n_1 n_2 n_3} \delta(\epsilon - \epsilon(n_1, n_2, n_3)) \]

where

\[ \epsilon(n_1, n_2, n_3) = \frac{\hbar^2}{2n} \left( \frac{2\pi}{\hbar} \right)^2 \left[ n_1^2 + n_2^2 + n_3^2 \right] \]

is the energy states of a particle in a box of sides \(L\), obtain the usual expression for Density of States \(g(\epsilon)\) of a free particle in a volume \(V = L^3\). Hint, use the index \(n_r^2 = n_1^2 + n_2^2 + n_3^2\) and spherical polar coordinates.

3. Derive the expression for the specific heat \(C_V/N\) of a Bose gas given by Eq. (7.74) in the class notes. Begin with the expression for the energy or entropy of the Bose gas.

4. For a Fermi gas, derive the accurate expression for the specific heat at low temperature,

\[ C_V = \frac{\pi^2}{2} R \left( \frac{T}{T_F} \right) \]

given by (8.19) in the class notes.
The free-electron model of the conduction electrons in metals seems naive but is often successful. Among other things, it gives a reasonably good account of the compressibility for certain metals. This prompts the following question. You are given the number density \( n \) and the Fermi energy \( E \) of a non-interacting Fermi gas at zero absolute temperature, \( T = 0 \) K. Find the isothermal compressibility

\[
\kappa = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T,
\]

where \( V \) is volume, \( p \) is pressure.

**Hint:** Recall that \( pV = \frac{2}{3} E \), where \( E \) is the total energy.

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In the very early stages of the universe, it is usually a good approximation to neglect particle masses and chemical potential compared with \( kT \).

(a) Write down the average number and energy densities of a gas of non-interacting fermions in thermal equilibrium under these conditions. (You need not evaluate dimensionless integrals of order 1.)

(b) If the gas expands adiabatically while remaining in equilibrium, how do the average number and energy densities depend on the dimensions of the system?

Assume that the fermions are predominantly electrons and positrons when \( T \approx 10^{11} \) K in parts (c) and (d) below.

(c) Is the assumption made in (a) that the particles are non-interacting reasonable? Why? [Hint: What is the average coulomb interaction energy?]

- Positron charge = \( 1.6 \times 10^{-19} \) coulomb; Boltzmann's constant \( k = 1.38 \times 10^{-16} \) erg/K.

(d) If the interaction cross sections in the electron-positron gas are typically of order of magnitude of the Thompson cross section \( \sigma_T = 8 \pi r_0^2 / 3 \) (classical electron radius \( r_0 = 2.8 \times 10^{-13} \) cm), estimate the mean free time between collisions of the particles. If the expansion rate in part (b) \( \approx 10^4 \) sec\(^{-1}\), is the assumption that the gas remains in equilibrium reasonable? Why?

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Fermi gas. Consider an ideal Fermi gas whose atoms have mass \( m = 5 \times 10^{-24} \) grams, nuclear spin \( I = \frac{1}{2} \), and nuclear magnetic moment \( \mu = 1 \times 10^{-23} \) erg/gauss. At \( T = 0 \) K, what is the largest density for which the gas can be completely polarized by an external magnetic field of \( 10^5 \) gauss? (Assume no electronic magnetic moment)