Homework Set 7.

Problem 1. *Velocity and spin current density operators for the Rashba spin-orbit Hamiltonian*—In recent spintronics research literature [see, e.g., Phys. Rev. B **64**, 121202 (2001) and cond-mat/0406531] one of the major theoretical issues has been to define correctly and evaluate the velocity operator $\hat{\mathbf{v}} = (\hat{v}_x, \hat{v}_y, \hat{v}_z)$, as well as the related spin current density operator,

$$\hat{j}_i^k = \frac{\hbar}{4}(\hat{\sigma}_k \hat{v}_i + \hat{v}_i \hat{\sigma}_k),$$

in systems with spin-orbit interaction (here $i, k = x, y, z$). They are essential in computing the transport properties within different linear response frameworks (Kubo formula, semiclassical Boltzmann equation, ...).

Assuming that two-dimensional electron gas is modeled by the Rashba SO Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} \otimes \hat{I}_2 + \frac{\alpha}{\hbar}(\hat{p}_y \otimes \hat{\sigma}_x - \hat{p}_x \otimes \hat{\sigma}_y),$$

where $\hat{I}_2$ is the unit operator in the spin space (while $\hat{I}$ below is the unit operator in the orbital Hilbert space of an electron), evaluate the following:

(a) show that the velocity operator is given by

$$\hat{v}_x = \frac{\hat{p}_x}{m} \otimes \hat{I}_2 - \frac{\alpha}{\hbar}\hat{I} \otimes \hat{\sigma}_y$$
$$\hat{v}_y = \frac{\hat{p}_y}{m} \otimes \hat{I}_2 + \frac{\alpha}{\hbar}\hat{I} \otimes \hat{\sigma}_x$$

Hint: the velocity operator can be obtained from the Hamilton equations of motion $\hat{v}_x = \frac{\partial \hat{H}}{\partial \hat{p}_x}$ or the Heisenberg equations of motion $\hat{v}_x = d\hat{x}/dt = [\hat{x}, \hat{H}]/i\hbar$.

(b) Using the velocity operator obtained in (a), evaluate explicitly all nine components $\begin{pmatrix} \hat{j}_x^x & \hat{j}_x^y & \hat{j}_x^z \\ \hat{j}_y^x & \hat{j}_y^y & \hat{j}_y^z \\ \hat{j}_z^x & \hat{j}_z^y & \hat{j}_z^z \end{pmatrix}$ of the spin current density tensor for the same SO Hamiltonian (note that expression should be simplified using properties of the products of the Pauli matrices).

(c) [*advanced*] If we employ the second-quantized notation, then the spin density operator is

$$\hat{S}^k = \sum_{\sigma,\sigma'=\uparrow,\downarrow} \frac{\hbar}{2} \hat{\Psi}^\dagger_\sigma(\mathbf{r}) \hat{\sigma}^k_{\sigma\sigma'} \hat{\Psi}_{\sigma'}(\mathbf{r})$$

while the spin current operator is

$$\hat{\mathbf{j}}^k = -\frac{i\hbar}{4m} \Big(\sum_{\sigma,\sigma'=\uparrow,\downarrow} \hat{\Psi}_\sigma(\mathbf{r})^\dagger \nabla \hat{\Psi}_{\sigma'}(\mathbf{r}) \hat{\sigma}^k_{\sigma\sigma'} - \text{h.c.} \Big)$$

Here h.c. stands for hermitian conjugate of the first term and $\sigma, \sigma' = \uparrow, \downarrow$ (or in terms of the coordinates of the two-components field operators or Pauli matrices, $\sigma, \sigma' = 1, 2$). Show that this two quantities do not obey continuity equation (which is satisfied by the conventional charge and current density operators)

$$\frac{d\hat{S}^k}{dt} + \nabla \cdot \hat{\mathbf{j}}^k = \hat{F}^k$$

That is, find explicit expression for $\hat{F}^k$ in the above equation, thereby proving that it turns to be non-zero in system with Rashba SO coupling.