Homework Set 6.

Problem 1. *Spin Entanglement*—Two spins are prepared in the following state:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle_A \otimes \left(\frac{1}{2}|\uparrow\rangle_B + \frac{\sqrt{3}}{2}|\downarrow\rangle_B\right) + \frac{1}{\sqrt{2}}|\downarrow\rangle_A \otimes \left(\frac{\sqrt{3}}{2}|\uparrow\rangle_B + \frac{1}{2}|\downarrow\rangle_B\right).$$

For this state, find the following:

(a) The von Neumann entropy $S = -\mathrm{Tr}\,[\hat{\rho}\log_2\hat{\rho}]$ of the reduced density matrices $\hat{\rho}_A = \mathrm{Tr}_B\,(|\Phi\rangle\langle\Phi|)$ and $\hat{\rho}_B = \mathrm{Tr}_A(|\Phi\rangle\langle\Phi|)$ which describe quantum states of subsystems A and B, respectively.

(b) The Schmidt decomposition for this state $|\Phi\rangle = \sum_i \sqrt{p_i}|u_i\rangle \otimes |v_i\rangle$. Hint: For a given state $|\Phi\rangle$ one can find (non-unique) special orthnonormal basis $|u_i\rangle$ in $\mathcal{H}_A$ and $|v_i\rangle$ in $\mathcal{H}_B$ so that the state is represented in a simplest form—as a sum of biorthogonal terms. Technically, p_i are the non-zero eigenvalues of the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$, while $|u_i\rangle$ and $|v_i\rangle$ are the corresponding eigenvectors [pedagogical reference: A. Ekert and P. L. Knight, Am. J. Phys. **63**, 415 (1995)].

Problem 2. *Resonant Spin Manipulation*—A quantum state of spin is described by two-component spinor

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix},\ \psi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which evolves (in the Schrödinger picture) due to the time-dependent Hamiltonian $\hat{H} = \hbar B_z\hat{\sigma}_z + B_R\cos(\Omega t)\hat{\sigma}_x$ [$(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the standard Pauli matrices with $\hat{\sigma}_z$ being the diagonal one].

(a) Show that $\exp(iB_z\hat{\sigma}_z t) = \cos(B_z t) + i\hat{\sigma}_z\sin(B_z t)$.

(b) Make change of variables $\psi(t) = \exp(-iB_z\hat{\sigma}_z t)\phi(t)$ and derive the equation which describes the time evolution of $\phi(t)$.

(c) Consider the resonance $B_z = \Omega/2$, solve this equation using the first order perturbation theory in B_R, and show when this perturbation theory breaks down.