Homework Set 5.

Problem 1. If $|\alpha\rangle$ is a state vector of a quantum system, then what is the value of the following expression $\mathrm{Tr}\,(|\alpha\rangle\langle\alpha|)$ (i.e., trace over states in the Hilbert space of the quantum system).

Problem 2. Suppose a sudden electric field pulse is applied to a material, $\vec{E} = \vec{E}_0\delta(t)$. Sketch qualitatively, as a function of time, the current $j(t)$ that would develop in

(a) metal

(b) insulator

(c) superconductor

Label the time scales in your sketch, both in qualitative and quantitative terms.

Problem 3. A spin state of an electron is described by the following density matrix:

$$\hat{\rho}_s = \frac{1}{3}\begin{pmatrix} 2 & 1+i \\ 1-i & 1 \end{pmatrix},$$

which is represented here in the basis $|\uparrow\rangle$, $|\downarrow\rangle$ of the eigenstates of $\hat{\sigma}_x$ (that is, $\hat{\sigma}_x|\uparrow\rangle = +|\uparrow\rangle$, $\hat{\sigma}_x|\downarrow\rangle = -|\downarrow\rangle$).

In order to characterize this state, examine the following properties:

(a) Find explicit representation of the Pauli operators in this basis, in which $\hat{\sigma}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal.

(b) What are the expectation values of the spin operator $\langle\vec{S}\rangle = \mathrm{Tr}\,(\hat{\rho}_s\hat{\vec{S}})$ where spin-$\frac{1}{2}$ operator is defined in terms of the Pauli operators $(\hat{S}_x,\hat{S}_y,\hat{S}_z)=\frac{\hbar}{2}(\hat{\sigma}_x,\hat{\sigma}_y,\hat{\sigma}_z)$? Note that $\langle\vec{S}\rangle \equiv \frac{\hbar}{2}\vec{P}$ where $\vec{P}$ is the Bloch vector, so this calculation should yield the direction and the intensity of $\vec{P}$ inside the Bloch sphere.

(c) Is this spin in a pure or in a mixed quantum state?