# Midterm exam: Solutions

**Problem 1.** This is essentially proof, by construction, of Reisz theorem. In finite dimensional vector space $U$ (dim $U = n$, to avoid the convergence issues when dealing with infinite-dimensional Hilbert space), functional $F$ acts on an arbitrary vector $|\phi\rangle$ expanded in orthonormal basis $|e_i\rangle_{i=1}^n$ in the following way:

$$|\phi\rangle = \sum_{i=1}^{n} \phi_n |e_i\rangle \tag{1}$$

$$F(|\phi\rangle) = \sum_{i=1}^{n} \phi_n F(|e_i\rangle), \tag{2}$$

which follows from its linearity properties. If we construct the following vector:

$$|f\rangle = \sum_{i=1}^{n} [F(|e_i\rangle)]^* |e_i\rangle, \tag{3}$$

then scalar product $(|f\rangle, |\phi\rangle) = \sum_{i=1}^{n} F(|e_i\rangle)\phi_n = F(|\phi\rangle)$ reproduces the action of the functional $F$ in Eq. (2). Thus, a one-to-one correspondence is established between vectors $F$ in the dual vector space $U^*$ of functionals (*bras*) and vectors (*kets*) $|f\rangle \in U$, that is independent basis independent (invariant antilinear 'isomorphism').

**Problem 2.**

- (a) meaningless;
- (b) scalar (square of a scalar product);
- (c) operator (outer product of two vectors);
- (d) scalar (expectation value of an operator
- (e) bra-vector

- (f) meaningless in a single Hilbert space
- (g) the same as (f)
- (h) operator (element of operator algebra)
- (i) scalar; after choosing a complete orthonormal set of vectors $|e_i\rangle$, we get

$$\begin{aligned}\mathrm{Tr}\,(|\alpha\rangle\langle\beta|) &= \sum_i \langle e_i|\alpha\rangle\langle\beta|e_i\rangle = \sum_i \langle\beta|e_i\rangle\langle e_i|\alpha\rangle \\ &= \langle\beta|\left(\sum_i |e_i\rangle\langle e_i|\right)|\alpha\rangle = \langle\beta|\alpha\rangle,\end{aligned}$$

  due to the completeness relation $\sum_i |e_i\rangle\langle e_i| = \hat{I}$.

Solutions to Problems 3, 4 and 5 are given in the pages to follow using Maple.

# Midterm PHYS 607, Problems 3, 4, 5

```
> with(LinearAlgebra):
```

## Problem 3: Projection Operators

```
> e_1:=1/sqrt(2)*<1,1,0>;
```

$$e\_1 := \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

```
> e_2:=1/sqrt(6)*<1,-1,2>;
```

$$e\_2 := \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$

```
> e_3:=1/sqrt(3)*<-1,1,1>;
```

$$e\_3 := \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$$

```
> P_1:=OuterProductMatrix(e_1,e_1);
```

$$P\_1 := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
> Multiply(P_1,P_1); #check that P_1*P_1=P_1
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
> P_2:=OuterProductMatrix(e_2,e_2);
```

$$P\_2 := \begin{bmatrix} \frac{1}{6} & \frac{-1}{6} & \frac{1}{3} \\ \frac{-1}{6} & \frac{1}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

```
> Multiply(P_2,P_2); #check that P_2*P_2=P_2
```

$$\begin{bmatrix} \frac{1}{6} & \frac{-1}{6} & \frac{1}{3} \\ \frac{-1}{6} & \frac{1}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

Since e_1, e_2, and e_3 are orthonormal, P_1+P_2+P_3=I

## Problem 4: Gram-Schmidt

```
> a_1:=<1,-1,1>;
```

$$a\_1 := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

```
> a_2:=<-1,0,1>;
```

$$a\_2 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

```
> a_3:=<2,-1,2>;
```

$$a\_3 := \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

```
> scalar_product:=(x,y)->Multiply(HermitianTranspose(x),y);
```

$$scalar\_product := (x, y) \rightarrow LinearAlgebra{:-}Multiply(LinearAlgebra{:-}HermitianTranspose(x), y)$$

```
> b_1:=a_1/sqrt(scalar_product(a_1,a_1));
```

$$b\_1 := \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$$

```
> b_2:=a_2-b_1*scalar_product(b_1,a_2);
```

$$b\_2 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

```
> b_2:=b_2/sqrt(scalar_product(b_2,b_2));
```

$$b\_2 := \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

```
> b_3:=a_3-b_1*scalar_product(b_1,a_3)-b_2*scalar_product(b_2,a_3);
```

$$b\_3 := \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

```
> b_3:=b_3/sqrt(scalar_product(b_3,b_3));
```

$$b\_3 := \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

```
> ord := GramSchmidt([a_1,a_2,a_3],normalized); #Check using
  internal Maple GS procedure
>
```

$$ord := \left[ \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \right]$$

```
> M:=<a_1|a_2|a_3>;
```

$$M := \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

```
> c:=<0,0,0>;
```

$$c := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> LinearSolve(M,c,free='s'); #extra check that these three vectors
  are indeed linearly independent (not needed on the exam)
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

# Problem 5: Change of operator representation (in non-orthonormal basis)

```
> T:=<<0,1,-1>|<1,0,-1>|<1,-1,0>>;
```

$$T := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

```
> Multiply(T,<0,1,-1>);
```

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

```
> Multiply(T,<1,-1,1>);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> Multiply(T,<-1,1,0>);
```

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

```
> T_in_other_basis:=<<1,0,0>|<0,0,0>|<0,0,-1>>;
```

$$T\_in\_other\_basis := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> S:=<<0,1,-1>|<1,-1,1>|<-1,1,0>>; #check using basis transformation matrix
>
```

$$S := \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

```
> MatrixInverse(S)&*T&*S;
```

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \&* \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}\right) \&* \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

```
> evalm(%);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**THE END**