Midterm exam, October 14

Problem 1. Establish a one-to-one correspondence between unitary vector space $|\phi\rangle \in U$ and its dual U^* (vector space of functionals $F(|\phi\rangle)$) by constructing explicitly a fixed vector $|f\rangle$ such that $F(|\phi\rangle) = (|f\rangle, |\phi\rangle)$ where $(\ ,\ )$ is a scalar product in the space U.

Problem 2. Let α and β be possible quantum states of a quantum system, while $\hat{A}$ is an operator in its Hilbert space. Which of the following expressions are allowed in the **bra-ket** formalism?

(a) $\langle\alpha\rangle$; (b) $\langle\alpha|\beta\rangle^2$; (c) $|\alpha\rangle\langle\beta|$; (d) $\langle\hat{A}\rangle$; (e) $\langle\alpha|\hat{A}$; (f) $|\alpha\rangle|\beta\rangle$; (g) $|\alpha\rangle^2$; (h) $\hat{A}^2$; (i) $\mathrm{Trace}\,(|\alpha\rangle\langle\beta|)$.

For each meaningful expression, state whether it is a vector, an operator, or a scalar. If (i) is meaningful, evaluate it (using *matrix* representation).

Problem 3. Consider three orthonormal vectors in $\mathbb{R}^3$ given by:

$$|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix},\ |e_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix},\ |e_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

What are the projection operators $\hat{P}_1$ and $\hat{P}_2$ projecting onto *span*$\{|e_1\rangle\}$ and *span*$\{|e_2\rangle\}$, respectively. What is $\hat{P}_1 + \hat{P}_2 + \hat{P}_3$?

Problem 4. Use the Gram-Schmidt process to find an orthonormal set of vectors out of (a) $(1,-1,1)$, $(-1,0,1)$, and $(2,-1,2)$. (b) Are these three vectors linearly independent? If not, find a zero linear combination of them by using part (a).

Problem 5. If the matrix representation of a linear operator $\hat{T}$ of $\mathbb{C}^3$ with respect to the standard basis is

$$\mathsf{T} = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{pmatrix},$$

what is the representation of $\hat{T}$ with respect to the basis $(0,1,-1)$, $(1,-1,1)$, $(-1,1,0)$?