

Homework Set 7.

Problem 1. Repeat problem 2. from the Set 5. by paper-and-pencil method, i.e., show that

$$(a) \text{Tr}[(\gamma \cdot a)(\gamma \cdot b)] = 4a \cdot b,$$

$$(b) \text{Tr}[(\gamma \cdot a)(\gamma \cdot b)(\gamma \cdot c)] = 0,$$

$$(c) \text{Tr}[(\gamma \cdot a)(\gamma \cdot b)(\gamma \cdot c)(\gamma \cdot d)] = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)],$$

by using linear functional properties of the trace and trace of the product of matrices, as well as fundamental anticommuting properties of the Dirac matrices $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{I}_4$, where metric tensor in the Minkowski space has components $g^{00} = 1 = -g^{11} = g^{22} = g^{33}$ (and all other components are zero) and $\mathbb{I}_4 = \text{diag}(1, 1, 1, 1)$ is a 4×4 unit matrix. Note that one way to show (b) is to first prove the following identity

$$\gamma_\lambda \gamma_\mu \gamma_\nu = g_{\lambda\mu} \gamma_\nu - g_{\lambda\nu} \gamma_\mu + g_{\mu\nu} \gamma_\lambda + i \varepsilon_{\lambda\mu\nu\rho} \gamma^\rho \gamma_5,$$

with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\varepsilon_{0123} = -1 = -\varepsilon^{0123}$ (other values follow from total antisymmetric properties of Levi-Civita tensor), and implied summation over indices that appear both as a subscript and superscript within the same term.

Problem 2. Find the eigenvalues and the corresponding orthonormal eigenvectors of the matrix:

$$A = \begin{pmatrix} 1 & \sqrt{8} & 0 \\ \sqrt{8} & 1 & \sqrt{8} \\ 0 & \sqrt{8} & 1 \end{pmatrix},$$

Using its eigenvalues and eigenvectors: (a) show that $A = \sum_i \lambda_i P_i$ where P_i is the projection matrix (representation of the projection operator) onto eigensubspace of eigenvalue λ_i (a subspace of \mathbb{C}^3 that is spanned by corresponding eigenvectors); (b) Diagonalize each projection matrix P_i . What is the correspondence between eigenvalues of P_i , $\text{Tr} P_i$, and the dimension of an eigensubspaces onto which it projects? Find representation of P_i in the basis of eigenvectors of A ; (b) By inspecting eigenvalues determine if this matrix has an inverse. If it does, find A^{-1} using the spectral decomposition from (a). Verify explicitly your finding by computing $A \cdot A^{-1}$.

Problem 3. Show that the determinant and the trace of a matrix, which is only infinitesimally different from the unit matrix, are connected in the following way: $\det(1 + \epsilon A) = 1 + \epsilon \text{Tr} A$, for $\epsilon \rightarrow 0$.