Homework Set 4.

Problem 1. In various electrodynamics problems one encounters Laplacian operator ∇^2 acting on functions $\{\phi(\mathbf{x})\}$ which are defined in some finite or infinite volume of space. What boundary conditions should be imposed on these functions in order that ∇^2 be Hermitian operator?

Problem 2. Let $|a_1\rangle = (1, 1, -1)$ and $|a_2\rangle = (-2, 1, -1)$.

(a) Construct (in the form of a matrix) the projection operators $\hat{P}_1$ and $\hat{P}_2$ that project on the direction of $|a_1\rangle$ and $|a_2\rangle$, respectively. Verify that they are indeed projection operators.

(b) Construct (in the form of a matrix) the operator $\hat{P} = \hat{P}_1 + \hat{P}_2$ and verify directly that it is a projection operator.

(c) Let $\hat{P}$ act on an arbitrary vector (x, y, z). What is the dot (i.e., scalar) product of the resulting vector with the vector $|a_1\rangle \times |a_2\rangle$? What can you say about $\hat{P}$ and your conclusion in (b)?

Problem 3. Find the matrix representation of a derivative operator $\hat{D}x(t) = \sum_{k=1}^{2} k\alpha_k t^{k-1}$, acting on the vectors $|x(t)\rangle = \sum_{k=0}^{2} \alpha_k t^k$ of space $\mathcal{P}_3^c[t]$, in the basis: (a) standard $\{1, t, t^2\}$ and (b) orthonormal with respect to scalar product $\langle x|y\rangle = \int_{-1}^{1} x^*(t)y(t)\,dt$ (*hint*: use the results of Homework Set 2.).

Problem 4. Prove the Baker-Campbell-Hausdorff formula for operators (or matrices as a special case of operators acting in $\mathbb{R}^n$ or $\mathbb{C}^n$):

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + [\hat{A}, \hat{B}]\alpha + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]]\alpha^2 + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]]\alpha^3 + \ldots$$

where exponential of an operator is formally defined by $e^{\hat{A}} = \sum_{k=0}^{\infty} \frac{\hat{A}^k}{k!}$ and α is a complex parameter.

Utilize BCH formula to evaluate $e^{i\hat{H}t/\hbar}\hat{X}e^{-i\hat{H}t/\hbar}$ where $\hat{X}$ is the position operator and $\hat{H} = \hat{P}^2/2m$ is the free-particle Hamiltonian, with $\hat{P}$ denoting the momentum operator. The commutator of $\hat{X}$ and $\hat{P}$ is $[\hat{X}, \hat{P}] = i\hbar$.