Homework Set 2.

Problem 1. Consider a linear transformation $\hat{T} : (\mathbb{R}^4, \mathbb{R}, +) \to (\mathbb{R}^3, \mathbb{R}, +)$:

$$\hat{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + x_3 - x_4 \\ x_1 + x_2 + 2x_3 + 2x_4 \\ x_1 - x_3 - 3x_4 \end{pmatrix}.$$

Find explicitly one basis set in `ker` $\hat{T}$. What is the the dimension of `ker` $\hat{T}$? What is the rank of $\hat{T}$?

Problem 2. Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

together with a unit matrix $\mathbb{I}$, form an orthogonal basis in the inner vector space of all complex 2×2 matrices [scalar product is defined by $\langle A, B \rangle = \mathrm{Tr}\,(A^\dagger B)$]. Find the "coordinates" (v_1, v_2, v_3, v_4) of a "vector" (representing the state of some qubit in a quantum computer) $\vec{v} = \begin{pmatrix} 0.4 & 1+i \\ 1-i & 0.6 \end{pmatrix}$ in the *orthonormal basis* $\{\mathbb{I}/\sqrt{2}, \sigma_x/\sqrt{2}, \sigma_y/\sqrt{2}, \sigma_z/\sqrt{2}\}$.

Problem 3. Prove the Cauchy-Schwarz inequality for $n \times n$ complex matrices:

$$|\mathrm{Tr}\,(AB)|^2 \leq \mathrm{Tr}(A^\dagger A)\mathrm{Tr}(B^\dagger B)$$

[hint: use the Cauchy-Schwarz inequality for vectors in $(\mathbb{C}^{n\times n}, \mathbb{C}, +)$].

Problem 4. Given the linearly independent vectors $x(t) = t^n$ for $n = 0, 1, 2, ...$ in $\mathcal{P}^{\mathbb{C}}[t]$ (vector space of polynomials over the field of complex numbers), use the Gram-Schmidt procedure to find the orhonormal polynomials $e_0(t)$, $e_1(t)$, and $e_2(t)$. The scalar (inner) product is defined as: (a) $\langle x(t), y(t) \rangle = \int_{-1}^{+1} dt\, x^*(t) y(t)$; or (b) with a nontrivial weight function, $\langle x(t), y(t) \rangle = \int_{-\infty}^{+\infty} dt\, e^{-t^2} x^*(t) y(t)$ (hint: for help with integrals in (b) check the course web site).