# Homework Set 10.

**Problem 1.** A sphere of radius $R$ encloses a homogeneous charge distribution characterized by the charge density $\rho$. Using the formalism of Maxwell tensor $\hat{\mathcal{T}}_M = \{\epsilon_0 \vec{E}, \vec{E}\} + \{\vec{B}/\mu_0, B\} - (\epsilon_0 E^2/2 + B^2/2\mu_0)\hat{I}$, find the force $\vec{F} = \int \hat{\mathcal{T}}_M \cdot d\vec{S}$ by which one half-sphere interacts with the other one.

**Problem 2.** Let $J$ be the 2m×2m skew-symmetric matrix $J = \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix}$, where $\mathbb{I}_m$ denotes the m×m unit matrix. Show that the set of all 2m×2m complex matrices $A$ satisfying $A^T J A = J$ forms a group (the so-called symplectic group) under matrix multiplication.

**Problem 3.** A *vector operator* is defined as $(\hat{V}^1, \hat{V}^2, \hat{V}^3)$, a set of three operators satisfying the following commutation relations with angular momentum: $[\hat{V}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{V}_k$. The algebra of angular momentum is determined by the standard commutation relations, $[\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk} J_k$, which is also demonstrating that, according to this definition, $(\hat{J}^1, \hat{J}^2, \hat{J}^3)$ is a vector operator. Show that $\hat{V}^i\hat{V}_i = \sum_{i=1}^3 \hat{V}^i\hat{V}_i$ (i.e., summation over the repeated upper and lower indices is assumed) commutes with all components of the angular momentum, and is therefore a scalar operator.

**Problem 4.** Tho particles with spin $s = 1/2$ have a Hamiltonian:

$$\hat{H} = \alpha \hat{S}_{1z} \otimes \hat{I} + \beta \hat{I} \otimes \hat{S}_{2z} + \gamma \vec{S}_1 \cdot \vec{S}_2$$

Here $\vec{S}_1 = (S_{1x}, S_{1y}, S_{1z}) = \hbar/2\vec{\sigma}_1$ is a spin operator for the first particle (and correspondingly, $\vec{S}_2$ is the spin operator for the second particle) where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. Find eigenvalues and eigenvectors of this system by starting from any representation that you find appropriate.

**Problem 5.** Two qubits in a quantum computer have become entangled due to unitary quantum evolution induced by some quantum logic gate—their state is described by the following vector (the so-called Bell state)

$$|\Psi\rangle = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}.$$

in the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of 2-dimensional vector spaces of each qubit ($|0\rangle$ and $|1\rangle$ are basis vectors in this space, corresponding to 0 and 1 of the Boolean logic of a classical computer, while infinitely many superpositions $\alpha|0\rangle + \beta|1\rangle$ lead to "quantum parallelism" in non-classical computers). Find a quantum state of the first qubit, i.e., the reduced statistical operator $\hat{\rho}_1 = \mathrm{Tr}_2 |\Psi\rangle\langle\Psi|$ (obtained by tracing over the states in the vector space of a second qubit) and its von Neumann entropy $S = -\mathrm{Tr}\, \hat{\rho} \ln \hat{\rho}$. What are these quantities for the second qubit? Repeat the same analysis if qubits are entangled into the Einstein-Podolsky-Rosen state $|\Psi\rangle = \frac{|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle}{\sqrt{2}}$.