Final Exam, December 20

Problem 1. Consider a physical system (e.g., a qubit) described in a two-dimensional vector space spanned by orthonormal basis vectors $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this basis, the Hamiltonian $\hat{H}$ and the two observables $\hat{A}$ and $\hat{B}$ are given by:

$$\hat{H} = \frac{\hbar\omega_0}{3}\begin{pmatrix} 5 & \sqrt{2}i \\ -\sqrt{2}i & 4 \end{pmatrix} \; \hat{A} = \frac{a}{3}\begin{pmatrix} 2\sqrt{2} & -i \\ i & -2\sqrt{2} \end{pmatrix} \; \hat{B} = \frac{b}{3}\begin{pmatrix} 1 & -4\sqrt{2}i \\ 4\sqrt{2}i & 5 \end{pmatrix},$$

where ω_0, a, and b are positive real constants. The physical system at $t = 0$ is in the state

$$|\Psi(0)\rangle = \sqrt{\frac{2}{5}}|1\rangle + \sqrt{\frac{3}{5}}|2\rangle.$$

(a) At $t = 0$, the energy is measured. What values can be found and with what probabilities? What is the average value of the energy at $t = 0$?

(b) Suppose that instead we measure physics quantity described by $\hat{B}$ at $t = 0$. What results can be found and with what probabilities? What is the average value of the measurement?

(c) Calculate $|\Psi(t)\rangle$ resulting from the unitary time evolution $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$.

(d) What can you say about the product of the uncertainties $\Delta\hat{A}\Delta\hat{B}$ in the measurements of $\hat{A}$ and $\hat{B}$ at $t = 0$.

(e) What is a complete set of compatible (i.e., commuting) observables for this system?

Problem 2. Prove the *no-cloning* theorem [W. K. Wooters and W. H. Zurek, *A single quantum state cannot be cloned*, Nature (London) **299**, 802 (1982)] using an example from the joint Hilbert space $\mathcal{H} = \mathcal{H}_{\rm original} \otimes \mathcal{H}_{\rm copy}$ of two qubits ($\dim\mathcal{H}_{\rm original} = \dim\mathcal{H}_{\rm copy} = 2$): Show that it is *impossible* to find a unitary operator $\hat{U}_{\rm QCM}$ ("quantum copier machine") in this space that would allegedly act as:

$$\hat{U}_{\rm QCM} : |\Psi\rangle_{\rm original} \otimes |\phi_0\rangle \mapsto |\Psi\rangle_{\rm original} \otimes |\Psi\rangle_{\rm copy}, \; \forall|\Psi\rangle \in \mathcal{H}_{\rm original}$$

where $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is an unknown vector (α and β are not known) to be copied and $|\phi_0\rangle$ is a "blank" state of the copy. Use *reductio ad absurdum* method—assume that $\hat{U}_{\rm QCM}$ exist, and then demonstrate that this assumption is incompatible with fundamental properties of vector spaces and operators acting on them.

Problem 3. Electromagnetic plane wave impinges on a surface at an angle θ and reflects with coefficient of reflection R (i.e., the energy density of the reflected wave is

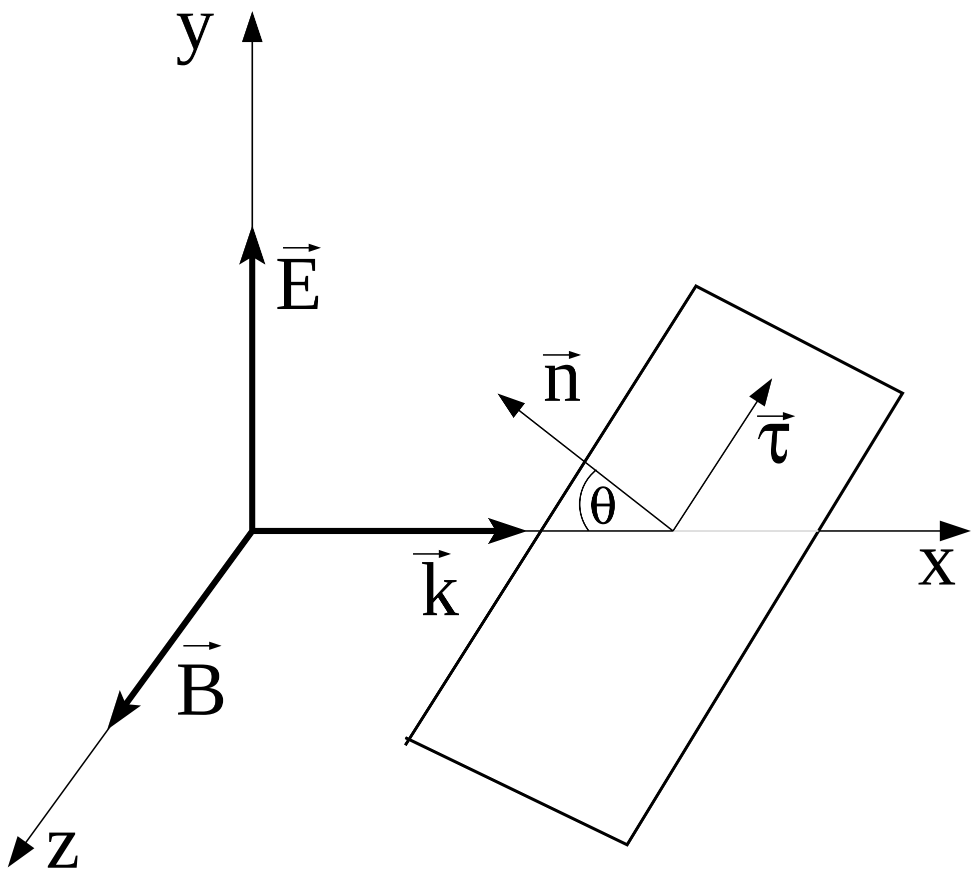

Figure 1: Relationship between different vectors in Problem 3: electrical field $\vec{E}$, magnetic field $\vec{B}$, direction of the electromagnetic wave propagation $\vec{k}$, normal vector to the surface $\vec{n}$, and tangential vector to the surface $\vec{\tau}$.

$w' = Rw$, where $w = \varepsilon_0 E^2/2 + B^2/2\mu_0$). Assuming that the wave propagates along the x-axis (see Figure 1; for electromagnetic wave in vacuum $|\vec{E}| = c|\vec{B}|$, where the speed of light is $c = 1/\sqrt{\varepsilon_0\mu_0}$), express the components of Maxwell tensor in this coordinate system. Using this result, find the normal F_n and the tangential F_τ components of the force that wave exerts on the unit area of the surface (F_n is called light radiation pressure, and was discovered experimentally by Lebedev in 1901).

Problem 4. Find electric $\vec{E} = -\nabla\phi - \frac{\partial A}{\partial t}$ and magnetic $\vec{B} = \operatorname{curl}\vec{A}$ field of a point charge q that is moving with constant velocity $\vec{v} = v\vec{e}_x$. The four vector of potential $A^\mu = (\frac{\phi}{c}, \vec{A})$ transforms from one reference frame to another via Lorentz transformations $A'^\mu = \Lambda^{\mu'}_{\alpha} A^\alpha$ ($\Lambda^{\mu'}_{\alpha} = \frac{\partial x'^\mu}{\partial x^\alpha}$ in the general tensor language), with inverse transformation being $A^\alpha = \Lambda^{\alpha}_{\mu'} A'^\mu$ ($\Lambda^{\alpha}_{\mu'} = \frac{\partial x^\alpha}{\partial x'^\mu}$). The relationship between matrices of the direct and inverse transform is $\Lambda^{-1} = g\Lambda^T g$, where $g = \mathrm{diag}(1, -1, -1, -1)$ is the matrix of a metric tensor and Λ^T is the transpose matrix of Λ. The Lorentz transformation between the two systems moving along common $x - x'$ axis with

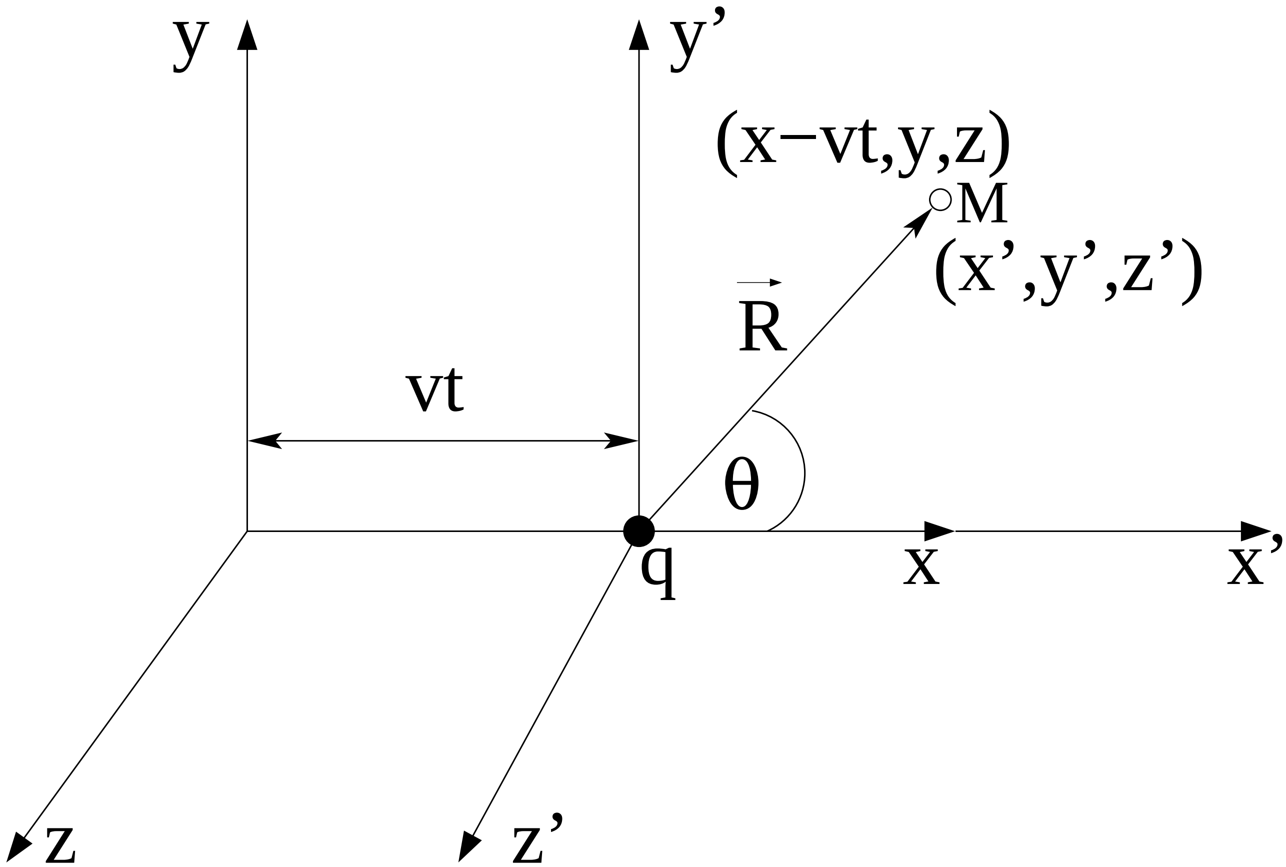

Figure 2: Relationship between reference frames in Problem 4 and corresponding coordinates of vector $\vec{R}$.

relative velocity $\vec{v}$ is,

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$. Express your result in terms of the relative position vector $\vec{R}$ of an arbitrary point M with respect to the particle and angle θ between $\vec{v}$ and $\vec{R}$ (see Figure 2). How does the electric field of the moving point charge look like in the ultrarelativistic limit $v/c \to 1$ (sketch your answer graphically)?