Homework Set 9.

Problem 1. Show that any operator $\hat{A}$ on a finite-dimensional complex inner product space can be written as $\hat{A} = \hat{U}\hat{R}$ where $\hat{U}$ is unitary and $\hat{R}$ is unique positive operator, respectively (note that if $\hat{A}$ is invertible, then $\hat{U}$ is also unique). Hint: start from $\hat{R} = \sqrt{\hat{A}^\dagger \hat{A}}$ (showing first that $\hat{A}^\dagger \hat{A}$ is a positive operator for arbitrary $\hat{A}$), and assume that $\hat{A}$ is invertible. Using this result, find the polar decomposition of a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -i \\ 1 & i & 0 \end{pmatrix}.$$

Problem 2. Matrices of rotations (such as rotations of a rigid body discussed below) in 3-dimensional (3D) space can be specified in terms of three parameters (i.e., relations $OO^T = O^T O = \mathbb{I}$ establish relations between nine matrix elements leaving only the three of them as independent). One choice of these three parameters are the so-called **Euler angles**, describing a general rotations as: a rotation of angle ϕ about the z-axis, followed by rotation of angle θ about the *new* x-axis, followed by a rotation angle ψ about the *new* z-axis. The rotation matrix $R(\phi, \theta, \psi)$ in terms of this angles is then:

$$\begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & -\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & -\cos\psi\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}.$$

Show that such matrix is indeed orthogonal. The other possible set of three parameters determining 3D rotation is the **axis of rotation** (which can be specified by two parameters) and the **angle of rotation** about that axis. Assuming that Euler angles are $\phi = 30^\circ$, $\theta = 45^\circ$, and $\psi = 60^\circ$, find a unit vector along such axis and the angle of rotation about the rotation axis.

Problem 3. Find the law of transformation of the tensor of inertia of a rigid body of mass $m = \sum_\alpha m_\alpha$

$$\mathcal{I}^O = \left(\sum_\alpha m_\alpha \rho_\alpha^2 \right) \hat{I} - \sum_\alpha \{ m_\alpha \vec{\rho}_\alpha, \vec{\rho}_\alpha \},$$

upon changing the origin of the coordinate system (rigidly) tied to the body from O to O' (here $\hat{I}$ is the usual unit operator in $\mathbb{R}^3$). Relationship between different coordinate systems and vectors with respect to them is shown in the Figure below. The formula for $\mathcal{I}^{O'}$ that you will get is called Huygens-Steiner theorem in theoretical

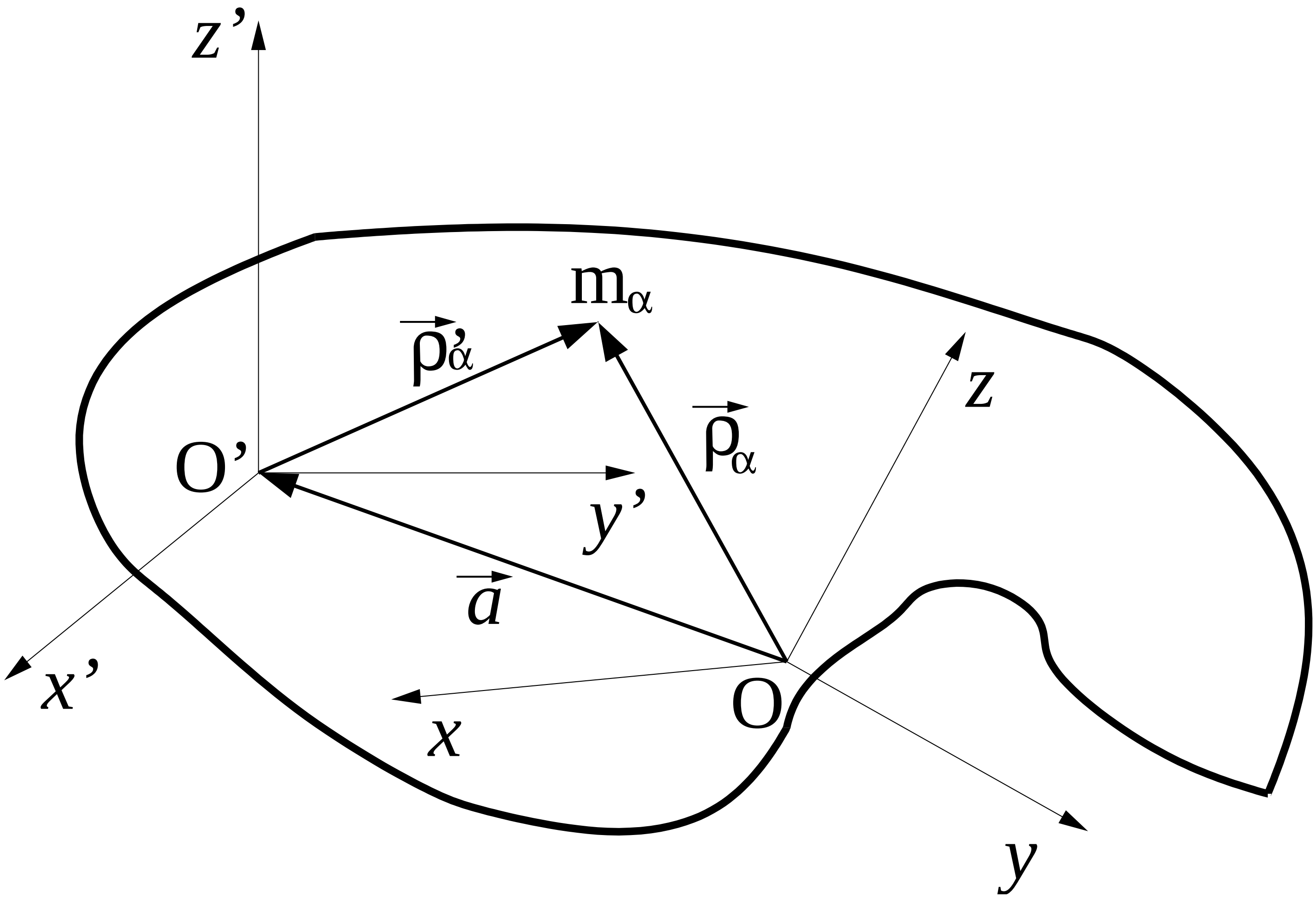

Figure 1: Relations between different vectors and coordinate systems in the Problem 3.

classical mechanics. Check the correctness of your finding by deriving the following two corollaries (which usually appear in the textbooks on CM):

(a) If $O' \equiv C$ is the center of mass of the body ($\vec{\rho}'_C = 0$, $\vec{a} = \vec{\rho}_C$) then $\mathcal{I}^O = \mathcal{I}^C + m(\rho_C^2 \hat{I} - \{\vec{\rho}_C, \vec{\rho}_C\})$.

(b) $I^O_{uu} = I^C_{uu} + m[\rho_C^2 - (\vec{u} \cdot \vec{\rho}_C)^2]$, where $I^O_{uu} = \vec{u} \cdot \mathcal{I}^O \cdot \vec{u}$ is the moment of inertia with respect to the axis determined by the unit vector $\vec{u}$ ($|\vec{u}| = 1$). This result, establishing a connection between two moments of inertia with respect to two different parallel axises passing through A and C, that are separated by a distance d (the expression in the brackets is obviously d^2), is usually found in introductory physics textbooks as the "Steiner theorem".

Problem 4. A certain rigid body may be represented by three point masses: $m_1 = 1$ at $(1, 1, -2)$, $m_2 = 2$ at $(-1, -1, 0)$ and $m_3 = 1$ at $(1, 1, 2)$, whose coordinates are given with respect to some chosen coordinate system . Find the matrix representation of the tensor of inertia in this coordinate system. Diagonalize this matrix obtaining the eigenvalues and the principal axes (determined by the the orthonormal eigenvectors) of the tensor.