Homework Set 8.

Problem 1. Find the square root $\sqrt{A}$ of the following matrix:

$$A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix},$$

Verify your result by showing that $\sqrt{A} \cdot \sqrt{A} = A$. Find also $\sin(\pi A/2)$.

Problem 2. Find the determinant of a $n \times n$ matrix A whose elements $A_{ii} = \beta$ and $A_{ij} = \alpha$ (for $i \neq j$) are real numbers:

$$\det A \equiv \begin{vmatrix} \beta & \alpha & \alpha & \ldots & \alpha \\ \alpha & \beta & \alpha & \ldots & \alpha \\ \alpha & \alpha & \beta & \ldots & \alpha \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha & \alpha & \alpha & \ldots & \beta \end{vmatrix} = F_n(\alpha, \beta).$$

Verify your general formula for $F_n(\alpha, \beta)$ [i.e., a function of α, β, and n] by comparing it with explicit calculation of the determinants of 2–, 3–, and 4–dimensional cases of A (*hint*: in the course of solving this problem you should encounter the so-called recurrence or difference equation for F_n; for a succinct and fast introduction to this topic check: `http://mathworld.wolfram.com/RecurrenceEquation.html`).

Problem 3. Consider the motion of a charged particle in a constant magnetic field pointing in the z-direction. The equation of motion for such particle is $m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$. Find the law of motion $(x(t), y(t), z(t))$ for this particle by: rewriting this equation in a matrix form $\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = M \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, solving the system of three differential equations in the representation in which M is diagonal (where they become uncoupled), and returning back to the original representation via basis transformation matrix. Initial velocity is (v_{0x}, v_{0y}, v_{0z}) while the initial position vector is (x_0, y_0, z_0).

Problem 4. Suppose that there are two operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A}, \hat{B}] = c\hat{\mathbb{I}}$, where c is a constant. Show that the vector space in which such operators are defined cannot be finite-dimensional. Conclude that the position and momentum operators of quantum mechanics can be defined only in infinite dimensional Hilbert space.

Problem 5. Show that an arbitrary matrix A can be "diagonalized" as $D = U \cdot A \cdot V$, where U is unitary and D is a real diagonal matrix with only nonnegative eigenvalues (*hint*: consider $A \cdot A^\dagger$). Note: this is one of the basic techniques in elementary particle physics to study mixing matrices of fermion (quarks and leptons) masses in the standard model Lagrangian, violation of symmetries (like CP), etc.