Homework Set 6.

Problem 1. Matrices U and A are related by $U = e^{i\alpha A}$, where α is a real parameter. Show that:

$$[U \text{ is unitary and det } U = 1] \Leftrightarrow A \text{ is Hermitian and } \mathrm{Tr}\,(A) = 0$$

Problem 2. In the current research in *spintronics* one has to describe injection, manipulation, and detection of electrons whose spin-polarization can point in various directions (determined by, e.g., by changing the orientation of magnetization of ferromagnetic sources and drains). In this context, the standard set of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

can be used to describe only electrons that are polarized along the z-axis (i.e., the standard basis of $\mathbb{C}^2$, that is by default basis in which these matrices are represented, is comprised of the two eigenvectors of σ_z). In fact, this set of 2×2 matrices is just one representation of the algebra of Pauli spin operators, where algebra multiplication is defined in a standard (via their commutator) way for any algebra of angular momentum operators, $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. Furthermore, the operators of the Pauli algebra satisfy the following special conditions:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1,$$

$$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z,\ \sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x,\ \sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y,$$

$$\sigma_x\sigma_y\sigma_z = i,$$

$$\mathrm{Tr}(\sigma_x)=\mathrm{Tr}(\sigma_y)=\mathrm{Tr}(\sigma_z)=0,\ \det(\sigma_x)=\det(\sigma_y)=\det(\sigma_z)=-1.$$

Starting from the representation of Pauli spin operators in which σ_z is diagonal, find two other representations (sets of 2×2 matrices) of this algebra: (a) where σ_x is diagonal matrix, and (b) where σ_y is diagonal matrix. Test your results by checking explicitly that these two sets (which rarely appear in textbooks) satisfy special properties listed above.