Homework Set 3.

Problem 1. For a given basis of vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ that span a crystal lattice one can define a reciprocal crystal lattice whose points are spanned by the reciprocal basis

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot (\vec{b} \times \vec{c})}, \ \vec{b}' = \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot (\vec{b} \times \vec{c})}, \ \vec{c}' = \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot (\vec{b} \times \vec{c})}.$$

Show that

- $\vec{x}' \cdot \vec{y} = \delta_{xy}$, $(\vec{x}, \vec{y} = \vec{a},\ \vec{b},\ \vec{c})$
- $\vec{a}' \cdot (\vec{b}' \times \vec{c}') = [\vec{a} \cdot (\vec{b} \times \vec{c})]^{-1}$
- $\vec{a} = \frac{\vec{b}' \times \vec{c}'}{\vec{a}' \cdot (\vec{b}' \times \vec{c}')}$

Problem 2. In relativistic quantum mechanics one can write Dirac equation in a compact form by using vector $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z) = \gamma_0 \vec{\gamma}$ whose components are α-matrices. They are defined in terms of 4×4 Dirac (or gamma)-matrices:

$$\gamma^0 = \gamma_0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \ \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

where $\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is 2×2 unit matrix and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices (introduced in the Homework Set 2.). Show that

$$\vec{\alpha} \times \vec{\alpha} = 2i\vec{\sigma},$$

by direct computation of corresponding cross (vector) product.

Note: Appendix to this homework set contains basic notation and properties of both Pauli and Dirac matrices (textbooks and research literature usually employ bold letters to denote 3-dimensional vectors, i.e., $\mathbf{a}$ and $\vec{a}$ label the same vector).