

Homework Set 11.

Problem 1. Electromagnetic field tensor is defined as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

which is *covariant curl* in the Riemann space (i.e., Minkowski space in this case) of a four vector of potential $A_\mu = (\frac{\phi}{c}, \vec{A})$ where ϕ is the scalar potential and \vec{A} is the vector potential of electromagnetic field.

(a) Write down explicitly the components of this tensor in the form of 4×4 anti-symmetric matrix. What is the corresponding matrix for $F^{\mu\nu}$?

(b) Particular contractions of this tensor are Lorentz invariants (or invariants of electromagnetic field): $I_1 = F_{\mu\nu}F^{\mu\nu}$ (scalar) and $I_2 = -\frac{c}{8}\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\rho\sigma}$ (pseudoscalar), where $\varepsilon^{\mu\nu\sigma\rho}$ is the total antisymmetric (pseudo)tensor Levi-Civita whose components are (as usual) 1 if $\mu\nu\sigma\rho$ is even permutation of 0123, -1 if it is odd, and 0 in other cases. Evaluate explicitly I_1 (scalar) and I_2 (pseudoscalar).

(c) Using the fact that $j^\mu = (c\rho, \vec{j})$ is a four vector current density (ρ is charge density and \vec{j} is ordinary current density from 3D space), prove that $F_{,\nu}^{\mu\nu} = -\mu_0 j^\mu$, which is a covariant formulation of the two Maxwell equation with the sources (the first and the fourth). In general, covariant derivative of a tensor $A^{\mu\nu}$ is a tensor of rank 3:

$$A_{,\sigma}^{\mu\nu} = \frac{\partial A^{\mu\nu}}{\partial x^\sigma} + \Gamma_{\sigma\rho}^\mu A^{\rho\nu} + \Gamma_{\sigma\rho}^\nu A^{\mu\rho}$$

where $\Gamma_{\sigma\rho}^\mu$ are Christoffel symbols that are equal to zero in the case of Minkowski space (which is flat).

Problem 2. Quantum field theory formalism (see, e.g., G. Serman, *Introduction to Quantum Field Theory*, Chapter I) is developed starting from a covariantly formulated Lagrangians. For example, for a free (with no sources) electromagnetic field $\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$, which is a Lorentz scalar since action $S = \int \int \int \int dV_4 \mathcal{L}$ must be a scalar, and volume element $dV_4 = dx^0 dx^1 dx^2 dx^3$ of Minkowski space is also scalar. From the principle of least action, $\delta S = \delta \int dt \int d^3\vec{x} \mathcal{L}(t, \vec{x}) = 0$, one can derive the Lagrange equations for electromagnetic field

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} = 0,$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Evaluate Lagrange equations for this system and express them in terms of free-space field strengths $\vec{E} = -\partial_0 \vec{A} - \nabla A_0$ and $\vec{B} = \nabla \times \vec{A}$. How many of Maxwell equations does this give, and why are the others also satisfied?

Problem 3. Calculate $\int_{-\infty}^{\infty} dx f(x)\delta(g(x))$, where f and g are two arbitrary functions and δ is the Dirac delta function. Prove all formulas that you use beyond the definition of δ function.