Final Exam: Solutions

Solution of Problem 1. (a) We first have to solve the eigenproblem of matrix in $\hat{H}$. Its eigenvalues are:

$$\begin{vmatrix} 5-\lambda & \sqrt{2}i \\ -\sqrt{2}i & 4-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 18 = (\lambda-3)(\lambda-6) = 0, \text{ so } \lambda = 3, 6$$

which means that the eigenvalues of $\hat{H}$ are $E_1 = \hbar\omega_0$ and $E_2 = 2\hbar\omega_0$. These are the only values that are obtained in measurement of energy. The *normalized* eigenvectors are (they are certainly orthogonal since they belong to different eigenvalues):

$$|E_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2}i \end{pmatrix}, \quad |E_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix},$$

. At $t = 0$, $|\Psi(0)\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix}$, so that probabilities of finding the system in eigenstates $|E_1\rangle$ or $|E_2\rangle$ at $t = 0$ are given through expectation values of the corresponding spectral projectors $\hat{P}_1 = |E_1\rangle\langle E_1|$ and $\hat{P}_2 = |E_2\rangle\langle E_2|$, respectively

$$\begin{aligned} p_1 &= \langle\Psi(0)|E_1\rangle\langle E_1|\Psi(0)\rangle = |\langle E_1|\Psi(0)\rangle|^2 = \left|\frac{\sqrt{2}-\sqrt{6}i}{\sqrt{15}}\right|^2 = \frac{8}{15}, \\ p_2 &= \langle\Psi(0)|E_2\rangle\langle E_2|\Psi(0)\rangle = |\langle E_2|\Psi(0)\rangle|^2 = \left|\frac{\sqrt{3}-2i}{\sqrt{15}}\right|^2 = \frac{7}{15}. \end{aligned}$$

The average energy at $t = 0$ in the state $|\Psi(0)\rangle$ is $\langle\Psi(0)|\hat{H}|\Psi(0)\rangle$:

$$\langle\hat{H}\rangle = \frac{\hbar\omega_0}{15}(\sqrt{2}\ \ \sqrt{3})\begin{pmatrix} 5 & \sqrt{2}i \\ -\sqrt{2}i & 4 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix} = \frac{22}{15}\hbar\omega_0$$

which is equal to $\hbar\omega_0\frac{8}{15} + 2\hbar\omega_0\frac{7}{15}$, as expected from the general probability theory.

(b) This is the same as (a), but for matrix $\hat{B}$ instead. Its eigenvalues are found from

$$\begin{vmatrix} 1-\lambda & -4\sqrt{2}i \\ 4\sqrt{2}i & 5-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 27 = (\lambda+3)(\lambda-9) = 0, \text{ so } \lambda = -3, 9$$

to be $B_1 = -b$ and $B_2 = 3b$. The normalized eigenvectors are found to be:

$$|B_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix}, \quad |B_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2}i \end{pmatrix},$$

The probabilities of finding the system at $t = 0$ in the eigenstates $|B_1\rangle$ and $|B_2\rangle$ of $\hat{B}$ are

$$\begin{aligned} q_1 &= \langle\Psi(0)|B_1\rangle\langle B_1|\Psi(0)\rangle = |\langle B_1|\Psi(0)\rangle|^2 = \frac{7}{15}, \\ q_2 &= \langle\Psi(0)|B_2\rangle\langle B_2|\Psi(0)\rangle = |\langle B_2|\Psi(0)\rangle|^2 = \frac{8}{15}. \end{aligned}$$

The average value of measurement of $\hat{B}$ at $t = 0$ in the state $|\Psi(0)\rangle$ is $\langle\Psi(0)|\hat{B}|\Psi(0)\rangle$:

$$\langle\hat{B}\rangle = \frac{b}{15}(\sqrt{2} \;\; \sqrt{3}) \begin{pmatrix} 1 & -4\sqrt{2}i \\ 4\sqrt{2}i & 5 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix} = \frac{17}{15}b,$$

which is equal to $-b\frac{7}{15} + 3b\frac{8}{15}$, as expected.

(c) We first have to find the evolution operator as a function of Hamiltonian $f(\hat{H})$ using the spectral decomposition of $\hat{H}$:

$$\hat{H} = E_1|E_1\rangle\langle E_1| + E_2|E_2\rangle\langle E_2| \Rightarrow \hat{U} = e^{-i\hat{H}t/\hbar} = e^{-iE_1t/\hbar}|E_1\rangle\langle E_1| + e^{-iE_2t/\hbar}|E_2\rangle\langle E_2|,$$

so that $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$ is given by

$$\begin{aligned} |\Psi(t)\rangle &= e^{-iE_1t/\hbar}|E_1\rangle\langle E_1|\Psi(0)\rangle + e^{-iE_2t/\hbar}|E_2\rangle\langle E_2|\Psi(0)\rangle \\ &= e^{-i\omega_0 t}\frac{1}{3\sqrt{5}}(\sqrt{2} - \sqrt{6}i) \begin{pmatrix} 1 \\ \sqrt{2}i \end{pmatrix} + e^{-i2\omega_0 t}\frac{1}{3\sqrt{5}}(\sqrt{3} - 2i) \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} \end{aligned}$$

(d) What can at least say that $\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle_0|$. It is easy to evaluate commutator of $\hat{A}$ and $\hat{B}$:

$$[\hat{A}, \hat{B}] = -4iab \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which means that

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle_0| = \frac{2ab}{5}\left|(\sqrt{2} \;\; \sqrt{3}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix}\right| = \frac{4\sqrt{6}ab}{5}.$$

(e) We know from (d) that $[\hat{A}, \hat{B}] \neq 0$. We also note that $|B_1\rangle = |E_2\rangle$ and $|B_2\rangle = |E_1\rangle$, so that $\hat{H}$ and $\hat{B}$ *do* have common eigenvectors and therefore must commute (i.e., we do not have to check that $|\hat{H}, \hat{B}] = 0$). In a strict terminological sense, $\hat{H}$ alone is already a complete set of commuting observables since it has a non-degenerate spectrum (and we can, therefore, also say that $\hat{B}$ is some function of $\hat{H}$).

Solution of Problem 2. The solution consists of writing *two lines* of possible actions of $\hat{U}_{\mathrm{QCM}}$:

1. $\hat{U}_{\mathrm{QCM}}|\Psi\rangle_{\mathrm{original}} \otimes |\phi_0\rangle = \hat{U}_{\mathrm{QCM}}(\alpha|0\rangle + \beta|1\rangle) \otimes |\phi_0\rangle = \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle$

which is a consequence of linearity (i.e., $\hat{U}_{\rm QCM}$ should copy each term in the linear superposition of two states). On the other hand, the right hand side of the alleged action of $\hat{U}_{\rm QCM}$ is

$$
\begin{aligned}
2.\ \hat{U}_{\rm QCM}|\Psi\rangle_{\rm original} \otimes |\phi_0\rangle &= |\Psi\rangle_{\rm original} \otimes |\Psi\rangle_{\rm copy} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\
&= \alpha^2|0\rangle \otimes |0\rangle + \alpha\beta|0\rangle \otimes |1\rangle + \beta\alpha|1\rangle \otimes |0\rangle + \beta^2|1\rangle \otimes |1\rangle.
\end{aligned}
$$

Obviously, these two expressions cannot be reconciled for arbitrary state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (i.e., for arbitrary α and β).

Solution of Problem 3. The Maxwell tensor $\hat{\mathcal{T}}^M = \{\epsilon_0\vec{E}, \vec{E}\} + \{\vec{B}/\mu_0, B\} - (\epsilon_0 E^2/2 + B^2/2\mu_0)\hat{I}$ of electromagnetic wave ($\varepsilon_0 E^2/2 = B^2/2\mu_0$ for EM waves) in a *given* coordinate system in Figure 1 has extremely simple form:

$$
\mathcal{T}^M = \begin{pmatrix} -w & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

where $w = \varepsilon_0 E^2/2 + B^2/2\mu_0$ is the energy density of the electromagnetic field. This follows from explicit evaluation of the components $\mathcal{T}^M_{ij} = \vec{e}_i \cdot \hat{\mathcal{T}}_M \cdot \vec{e}_j$ of its matrix, which are all equal to zero except for $\mathcal{T}^M_{xx} = -w$. The force per unit area is just $\mathcal{T}^M \cdot \vec{n}$ (if we want a force this expression would then be multiplied by the area A of the surface—there is no need to integrate anything since components of a tensor do not depend on position in space):

$$
\vec{F} = \frac{1}{A}\int \mathcal{T}^M \cdot \vec{n} dS = \mathcal{T}^M \cdot \frac{\int dS}{A} = \begin{pmatrix} -w & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = w\cos\theta\,\vec{e}_x
$$

where unit vector $\vec{n}$ orthogonal to the surface is expressed via its components in a given coordinate system in which we have previously represented Maxwell tensor. Thus, the result can be expressed in a basis independent form is we note that $\cos\theta = -\vec{n}\cdot\vec{e}_x = -\vec{n}\cdot\vec{\kappa}$

$$
\vec{F} = -w\vec{\kappa}(\vec{\kappa}\cdot\vec{n}),\ \vec{\kappa} = \frac{\vec{k}}{|\vec{k}|},
$$

which is physically transparent since wave exerts force in the direction of its propagation ($\vec{\kappa}$ is unit vector along this direction). Since wave is reflected, the full force is the sum of the forces of incident and reflected waves:

$$
\vec{F} = -w\vec{\kappa}(\vec{\kappa}\cdot\vec{n}) - w'\vec{\kappa}'(\vec{\kappa}'\cdot\vec{n}),
$$

where $\kappa' = \frac{\vec{k}'}{|\vec{k}'|}$, is unit vector in the direction of propagation of reflected wave (see Figure 1). From here we get for the normal force (using that $w' = Rw$):

$$
\vec{F}_n = \vec{F}\cdot\vec{n} = -w(\vec{\kappa}\cdot\vec{n})^2 - w'(\vec{\kappa}'\cdot\vec{n})^2 = -w\cos^2\theta - Rwcos^2\theta = -(1+R)w\cos^2\theta,
$$

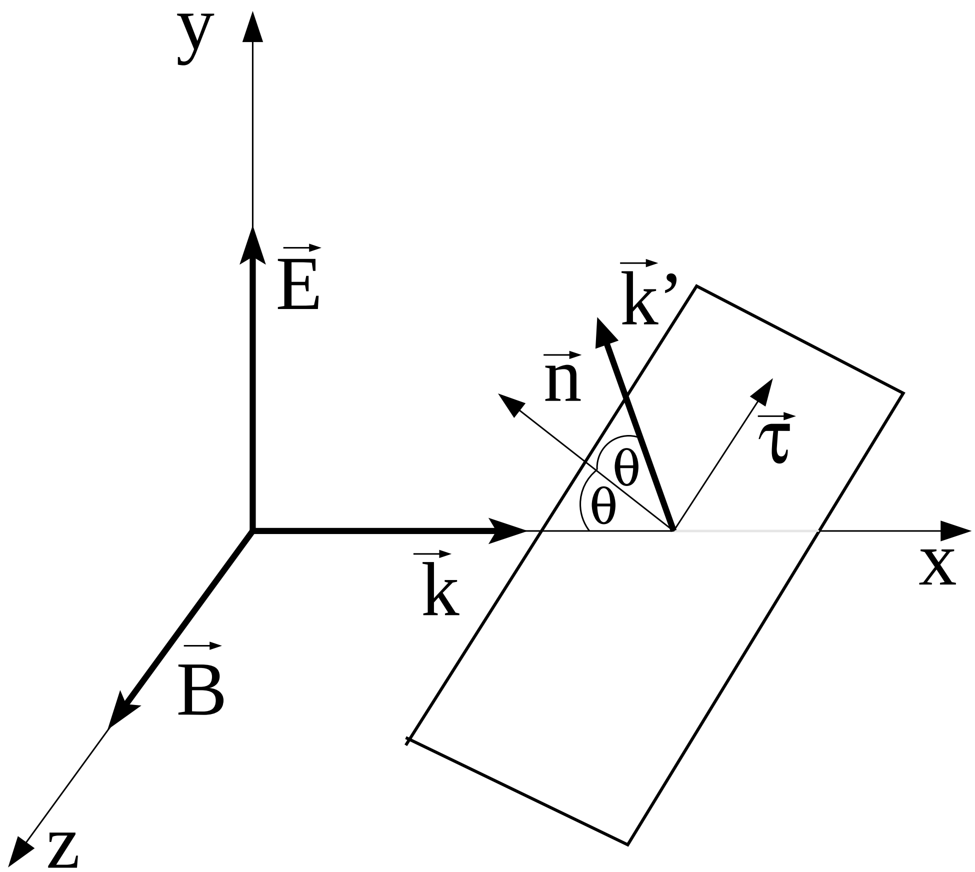

Figure 1: Relationship between different vectors in the solution Problem 3: electrical field $\vec{E}$, magnetic field $\vec{B}$, direction of the incident electromagnetic wave propagation $\vec{k}$ (defining a unit vector $\kappa = \vec{k}/|\vec{k}|$) and reflected wave propagation $\vec{k}'$ (defining a unit vector $\kappa' = \vec{k}'/|\vec{k}'|$) normal vector to the surface $\vec{n}$, tangential vector to the surface $\vec{\tau}$.

while the tangential force is

$$\begin{aligned}\vec{F}_\tau = \vec{F}\cdot\vec{\tau} &= -w(\vec{\kappa}\cdot\vec{\tau})(\vec{\kappa}\cdot\vec{n}) - w'(\vec{\kappa}'\cdot\vec{\tau})(\vec{\kappa}'\cdot\vec{n}) \\ &= -w\sin\theta(-\cos\theta) - Rw\sin\theta\cos\theta \\ &= (1-R)w\sin\theta\cos\theta.\end{aligned}$$

It is interesting to note that besides Lebedev radiation pressure F_n, there is always non-zero tangential component F_τ, unless $R = 1$ or $\theta \in \{0, \pi/2\}$.

Solution of Problem 4. For the observer in reference frame S' which is moving together with the particle, the charge is fixed and the field is just the *electrostatic* Coulomb field:

$$\phi' = \frac{1}{4\pi\varepsilon_0}\frac{q}{R'} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x'^2+y'^2+z'^2}},\ \ \vec{A}' = 0.$$

From $(\frac{\phi'}{c}, \vec{A}')$, we obtain the four vector of potential in the fixed laboratory frame

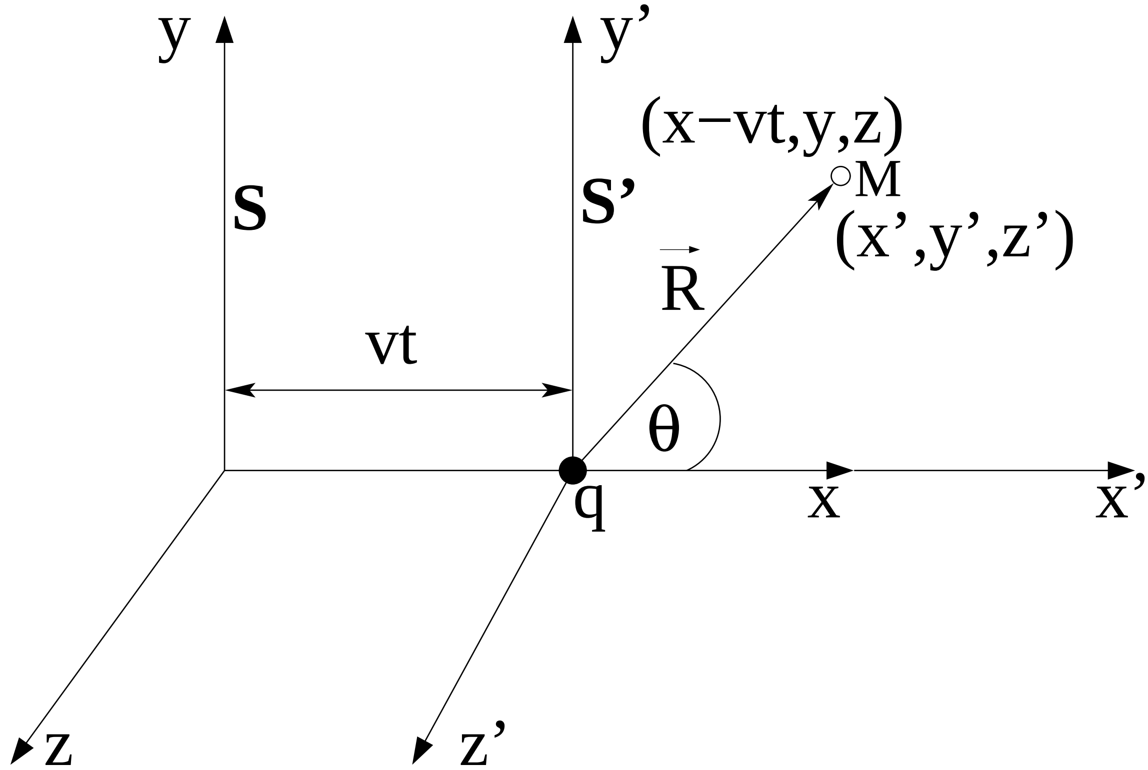

Figure 2: Relationship between reference frames in Problem 4 and corresponding coordinates of vector $\vec{R}$.

S via inverse Lorentz transformation of A'^{μ}:

$$\begin{pmatrix} \frac{\phi}{c} \\ A_x \\ A_y \\ A_z \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} \frac{\phi'}{c} \\ A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\phi'}{c} \\ A'_x \\ A'_y \\ A'_z \end{pmatrix}.$$

The form of Λ^{-1} can be easily obtained just from physical arguments that this is a transformation from system S' into a system S which is moving from the viewpoint of S' with velocity $-\vec{v}$ (so, $\beta \to -\beta$). Thus, the four vector of potential in the frame S is:

$$\begin{aligned} A^{\mu} &= \left(\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-v^2/c^2}}\frac{1}{\sqrt{\left(\frac{x-vt}{\sqrt{1-v^2/c^2}}\right)^2 + y^2 + z^2)}}, \frac{v}{c^2}\phi, 0, 0\right) \\ &= \left(\frac{q}{4\pi\varepsilon_0}\frac{1}{[(x-vt)^2 + (1-v/c)(y^2+z^2)]^{1/2}}, \frac{\vec{v}}{c^2}\phi\right) \end{aligned}$$

where we have used the fact that $x' = (x-vt)/\sqrt{1-v^2/c^2}$, $y' = y$, and $z' = z$. From the definition of electric $\vec{E} = -\nabla\phi - \frac{\partial A}{\partial t}$ and magnetic $\vec{B} = \operatorname{curl}\vec{A}$ field we now get:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{(1-v^2/c^2)\vec{R}}{R^{*3}}, \;\; \vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

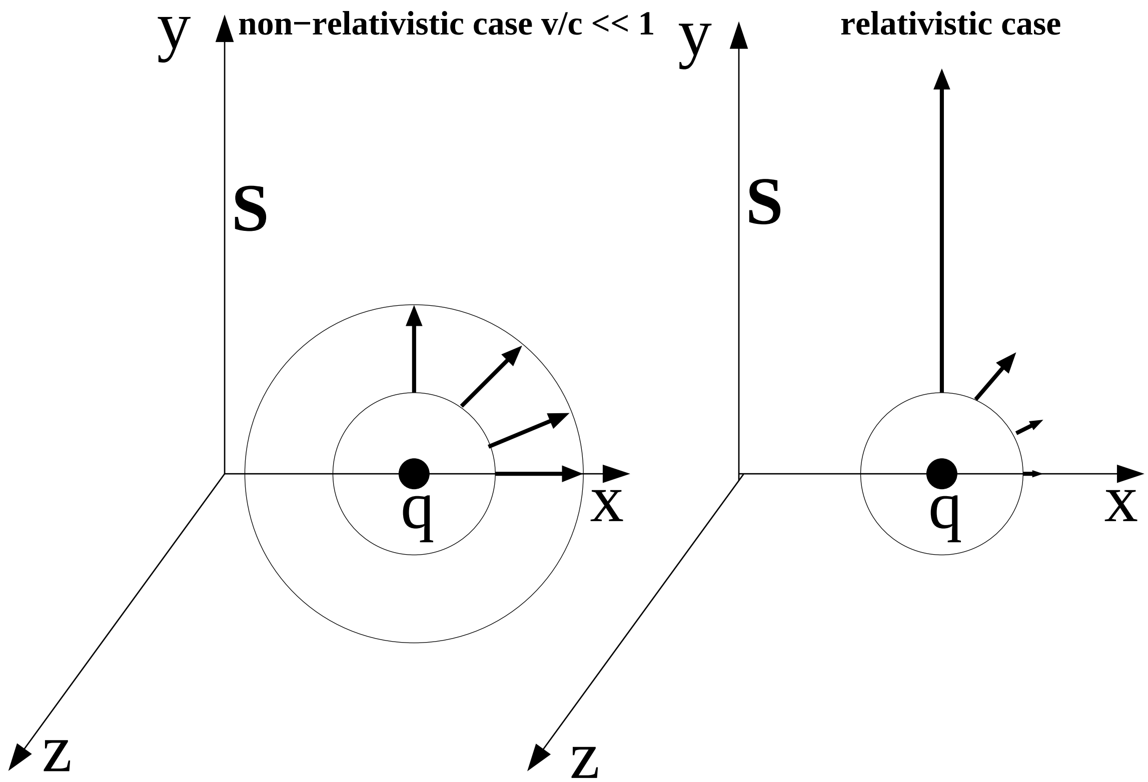

Figure 3: Electrical field in Eq. (1) for non-relativistic case $v/c \ll 1$ and relativistic case $v \sim c$.

with $R^* = \sqrt{(x-vt)^2 + (1-v^2/c^2)(y^2+z^2)}$. The electrical field can be recast in the following simple form,

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta/c^2)^{3/2}} \frac{\vec{R}}{R^3}, \tag{1}$$

using the fact that position vector of some point M with respect to the particle is different in the S and S' (as shown in Figure 2), such that

$$x - vt = R\cos\theta, \text{ and} \sqrt{y^2+z^2} = R\sin\theta \ \Rightarrow R^* = R\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}.$$

Physical meaning of this result is as follows (sketched in Figure 3): when $v/c \to 0$, the second factor in Eq. (1) is equal to one and the field is just ordinary Coulomb field around point particle, at each instant of time. As v/c increases, $\vec{E}$ has the greatest intensity for $\theta = \pi/2$, while the smallest intensity is for $\theta = 0$ or $\theta = \pi$ (i.e., along the trajectory of the particle). In the ultrarelativistic limit $v/c \to 1$, electrical field is different than zero only in the planes orthogonal to the direction of motion, i.e., in the angular interval $\Delta\theta = \sqrt{1 - v^2/c^2}$.