From Pythagoras to SUSY: Physics, Mathematics, and the Meaning of Mathematical Physics

PHYS 607 – Methods of Mathematical Physics
Instructor: Branislav K. Nikolić
C.N. Yang: Physics is not mathematics, just as mathematics is not physics. Somehow nature chooses only a subset of the very beautiful and complex and intricate mathematics that mathematicians develop and that precise subset is what the theoretical physicist is trying to look for.

Mathematics:
Rigorously proven theorems and purification of abstract concepts.

Physics:
Empirical problem solving and model building. Evade no-go theorems (e.g., SUSY).

P.A.M. Dirac: The most powerful method of advance [in physics] … is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and … to try to interpret the new mathematical features in terms of physical entities.
Efficiency of Abstraction (trivial)

- **Integers:** 1, 2, 3, ...
- **...** -3, -2, -1 and 0 and 1, 2, 3, ...
- **Rational:** p/q
- **Irrational:** $\sqrt{2}$ (Pythagoreans sacrifice 40 cows)
- **Real:** $[0, 1] \rightarrow \mathbb{R}$  
  \[ x \rightarrow \frac{1}{2}(1+\tanh x) \]
- **Complex:** $(\mathbb{R}, \mathbb{I}) \in \mathbb{R}^2$  
  \[ (0, 1)^2 = -1 \]
- **SUSY:** \( \{\chi_i, \chi_j\} = \chi_i \chi_j + \chi_j \chi_i = 0; \chi_i^2 = 0; \int d\chi_i = 0; \int \chi_i d\chi_i = 1 \)
  \[ x^\mu x^\nu - x^\nu x^\mu = 0; \quad x^\mu \chi_\alpha - \chi_\alpha x^\mu = 0; \quad \{\chi_\alpha, \chi_\beta\} = 0 \]
## Algebra versus Analysis in SM, QFT, and CMT

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Efficiency of Abstraction (advanced)

- Hilbert space theory enters Quantum Mechanics (von Neumann).
- Groups and group representations (Wigner and Weyl – originators of the ‘group theory disease’): \( CM: \vec{x}(t) \Rightarrow R\vec{x}(t) \); \( QM: |\Psi\rangle \rightarrow |\Psi\rangle + R|\Psi\rangle \).
- Manifolds in General Relativity (Kruskal’s extension of the Schwarzschild’s chart).
- Planck: “Let’s quantize the electromagnetic field” – turns out to be notoriously difficult – in the axiomatic approach fields are operator-valued distributions:

\[
\phi_f^a = \int d^4 x \; f(\vec{x}^\mu) \phi^a(\vec{x}^\mu); \; \phi_f : H \rightarrow H
\]

- Geometry of QFT (fiber bundles, gauge fields as connections, …)
- Quantum Gravity: Loops, M-theory (strings and branes), …
Symmetries in Physics: Wigner’s Legacy

Poincaré group

ISL (2,C)

Trivial representation

Massive states

Massless states

Symmetry group: special relativity (Lorentz + space-time translations)

Covering group

Irreducible representations

Vacuum

Spin J

2J +1 components

e.g. electrons, quarks

Spin J

1 component

Per helicity

e.g. neutrinos, photons, gravitons

\[ P^{(\mu\nu)} |\Psi\rangle = |\Psi\rangle \]